

COMPONENT REDUCTIONS AND THE CONNECTED RELATIONAL FIXED POINT PROPERTY

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ABSTRACT. We characterize the connected relational fixed point property (CRFPP) for finite nonempty posets, answering Problem 14 in Schröder’s *Ordered Sets* (second edition). For a point x and each connected component C_i of its incomparability set, the subposet Q_i consisting of C_i and all points comparable with x is a retract. We prove that P has CRFPP if and only if every Q_i does. Combined with deletion of irreducible points, this yields a recursive structural characterization. Equivalently, CRFPP fails precisely when P has a retract Q with more than one point such that Q has no irreducible points and every incomparability set of Q is nonempty and connected. We also obtain explicit witnesses for failure and homotopy invariance for finite T_0 spaces.

1. INTRODUCTION

On a finite nonempty partially ordered set, an *isotone relation* is a nonempty-valued multifunction that is monotone in both the upper and lower senses. The *connected relational fixed point property*, abbreviated CRFPP, requires a point x with $x \in \Phi(x)$ for every such multifunction whose values are connected. The property is formulated in [5, Exercise 4-51]. Schröder explicitly posed its characterization as an open problem [5, Problem 14, p. 110]. In this paper we resolve the problem by establishing a component-reduction theorem and two equivalent structural characterizations.

Related properties for isotone relations were studied by Walker [7] and Roddy–Schröder [2]. In particular, without the restriction to connected values, the finite relational fixed point property is characterized by successive deletion of irreducible points [5, Theorem 4.47]. Connected collapsibility implies CRFPP, but the converse fails [5, Exercises 4-51 and 7-5].

Retractions have long been used to study fixed point properties; reduction methods for the ordinary fixed point property include retractable subsets and deletions of one or two points [3, 4]. The retractions constructed below are instances of the collapse maps associated with *retractable sets* in the sense of Schröder [3]; see also [5, Exercise 4-21]. The feature particular to connected-valued isotone relations is a *localization equivalence*: the connectedness of the value at a chosen point identifies a component onto which a fixed-point-free

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relation may be projected without creating a new fixed point. This observation is the principal ingredient of the characterization. For context, multivalued fixed point theorems on finite spaces under different semicontinuity and acyclicity hypotheses appear in [1].

Throughout, all posets are finite and nonempty, and all subposets carry the induced order. We formulate the component reduction in Section 3, derive two structural characterizations in Sections 4 and 5, and discuss a decision procedure, explicit witnesses, and finite-space topology in Section 6.

2. PRELIMINARIES

For $x, y \in P$, write $x \parallel y$ when x and y are incomparable. Put

$$\begin{aligned} \text{Inc}_P(x) &= \{y \in P : y \parallel x\}, \\ N_P(x) &= \{y \in P : y \leq x \text{ or } x \leq y\} = P \setminus \text{Inc}_P(x). \end{aligned}$$

Thus $N_P(x) = \downarrow_P x \cup \uparrow_P x$, with both cones including x . Connectedness of a subset of P means connectedness in its *induced* comparability graph: two vertices are adjacent when they are distinct and comparable. Equivalently, any two elements can be joined by a fence inside the subset [5, Definition 2.43]. A singleton is connected; we never count the empty set as a connected value.

For nonempty $A, B \subseteq P$, the usual two-sided domination relation is

$$A \preceq_P B \iff \begin{cases} (\forall a \in A)(\exists b \in B) a \leq b, \\ (\forall b \in B)(\exists a \in A) a \leq b. \end{cases}$$

This relation is a preorder, not necessarily a partial order. A map $\Phi : P \rightarrow \mathcal{P}(P) \setminus \{\emptyset\}$ is an *isotone relation* when $x \leq y$ implies $\Phi(x) \preceq_P \Phi(y)$. Its fixed points are the elements $x \in \Phi(x)$. We write $\text{CRFPP}(P)$ when every isotone relation with nonempty connected values on P has a fixed point. These conventions coincide with [2, Definitions 2.1–2.2] and [5, Definitions 4.40–4.41 and Exercise 4-51].

A point $a \in P$ is *irreducible* when it has exactly one upper cover or exactly one lower cover. In a finite poset, this is equivalent to having a least strict upper bound or a greatest strict lower bound.

Lemma 2.1 (Comparable values). *Let Φ be an isotone relation on P . If Φ has no fixed point, then*

$$\Phi(x) \subseteq \text{Inc}_P(x) \quad (x \in P).$$

Proof. Suppose first that $x_1 \in \Phi(x_0)$ and $x_0 \leq x_1$. The first clause in the definition of $\preceq_P$ produces $x_{n+1} \in \Phi(x_n)$ with $x_n \leq x_{n+1}$ for all n . Finiteness forces equality at some stage, hence a fixed point. If $x_1 \leq x_0$, use the second clause to obtain a nonincreasing sequence instead. Thus a fixed-point-free relation cannot contain any element comparable with its argument. $\square$

The preceding fact is the finite case of [2, Lemma 2.8] and its dual. We also need two elementary properties of order-preserving maps.

Lemma 2.2 (Images and retracts). *Suppose $f : P \rightarrow R$ is order-preserving.*

- (i) *If $A \preceq_P B$ are nonempty subsets of P , then $f[A] \preceq_R f[B]$.*
- (ii) *If $A \subseteq P$ is connected, then $f[A]$ is connected.*

Consequently, if Q is a retract of P , then

$$\text{CRFPP}(P) \implies \text{CRFPP}(Q).$$

Proof. For (i), given $f(a) \in f[A]$, choose $b \in B$ with $a \leq b$; then $f(a) \leq f(b)$. Conversely, given $f(b) \in f[B]$, choose $a \in A$ with $a \leq b$. These prove the two required clauses. For (ii), the image of a comparability path is a walk with consecutive vertices equal or comparable; deleting repetitions shows that its image is connected.

Let $r : P \rightarrow Q \subseteq P$ be an order-preserving retraction, so that $r(q) = q$ for every $q \in Q$. Suppose that Q admits a fixed-point-free isotone relation Ψ with connected values. Define $\Phi(p) = \Psi(r(p)) \subseteq Q \subseteq P$. This has nonempty connected values; it is isotone because $p \leq q$ implies $r(p) \leq r(q)$ and hence $\Psi(r(p)) \preceq_P \Psi(r(q))$. A fixed point $p \in \Phi(p)$ would lie in Q and hence satisfy $p = r(p) \in \Psi(p)$, a contradiction. This proves the last assertion. $\square$

Lemma 2.3 (Irreducible-point deletion). *If a is irreducible in P , then*

$$\text{CRFPP}(P) \iff \text{CRFPP}(P \setminus \{a\}).$$

Proof. Let b be the unique upper or lower cover of a . The map $r : P \rightarrow Q := P \setminus \{a\}$ sending a to b and fixing all other elements is an isotone retraction. Thus $\text{CRFPP}(P)$ implies $\text{CRFPP}(Q)$ by Lemma 2.2.

Conversely, suppose Φ is a fixed-point-free isotone relation on P with connected values. For $q \in Q$ set $\Theta(q) = r[\Phi(q)]$. By Lemma 2.2, this is again an isotone relation with connected nonempty values. If $q \neq b$, then $r^{-1}(\{q\}) = \{q\}$, so $q \in \Theta(q)$ implies $q \in \Phi(q)$. If $q = b$, the only possible preimages are a, b , both comparable with b , and Lemma 2.1 rules out their presence in $\Phi(b)$. Hence Θ has no fixed points. By contraposition, $\text{CRFPP}(Q)$ implies $\text{CRFPP}(P)$. $\square$

We give a short proof of the finite incomparability lemma underlying Walker's characterization of the unrestricted relational fixed point property [7]; compare [5, Lemma 4.46].

Lemma 2.4 (Incomparability lemma). *Suppose $|P| > 1$ and P has no irreducible points. Then*

$$\Gamma_P(x) = \text{Inc}_P(x) \quad (x \in P)$$

is an isotone relation with nonempty values and no fixed points.

Proof. If some x were comparable with every point and were not maximal, choose an upper cover u of x . Comparability with x forces x to be the unique lower cover of u . If x were maximal, it would be the greatest element, and the dual argument would supply an irreducible point. Thus $\text{Inc}_P(x)$ is nonempty for all x .

For $p < q$, suppose $a \parallel p$. If no element at or above a were incomparable with q , then a would be comparable with q . Since $q \leq a$ would imply $p \leq a$, necessarily $a < q$. Choose a maximal z with $a \leq z < q$. Every strict upper bound w of z also lies at or above a and hence is comparable with q . Maximality excludes $w < q$, so $q \leq w$. Consequently q is the unique upper cover of z , a contradiction. Thus there exists $b \in \text{Inc}_P(q)$ with $a \leq b$, establishing the first clause of $\Gamma_P(p) \preceq_P \Gamma_P(q)$. The second clause follows by duality. The relation is fixed-point-free by definition. $\square$

3. COMPONENT LOCALIZATION

The key observation is that a connected value of a fixed-point-free isotone relation must lie in a single component of the incomparability set of its argument. The following theorem turns this observation into a reduction to retracts.

Theorem 3.1 (Component-reduction theorem). *Let P be a finite nonempty poset, let $x \in P$, and suppose that $\text{Inc}_P(x)$ has connected components $C_1, \dots, C_k$, with $k \geq 1$. For each i , let*

$$Q_i := N_P(x) \cup C_i.$$

Then Q_i is an induced retract of P and

$$\boxed{\text{CRFPP}(P) \iff \bigwedge_{i=1}^k \text{CRFPP}(Q_i).} \quad (1)$$

If $k \geq 2$, each Q_i is a proper subposet. If $\text{Inc}_P(x) = \emptyset$, then $\text{CRFPP}(P)$ holds.

Proof. For each i , define $r_i : P \rightarrow Q_i$ by

$$r_i(p) = \begin{cases} p, & p \in Q_i, \\ x, & p \notin Q_i. \end{cases}$$

We verify that r_i is isotone. For $p \leq q$, if both points lie in Q_i , or if both lie outside it, their images are ordered. If $p \notin Q_i$ and $q \in Q_i$, then p lies in a component of $\text{Inc}_P(x)$ other than C_i . The point q cannot belong to C_i : the comparison $p \leq q$ would otherwise join two distinct components of $\text{Inc}_P(x)$. Thus $q \in N_P(x)$. The possibility $q \leq x$ would imply $p \leq x$, contrary to $p \parallel x$. Hence $x < q$, and $r_i(p) = x \leq q = r_i(q)$. The remaining case $p \in Q_i$ and $q \notin Q_i$ is order dual: then $r_i(p) = p < x = r_i(q)$. Thus r_i is order-preserving and fixes Q_i pointwise.

By Lemma 2.2, $\text{CRFPP}(P)$ implies $\text{CRFPP}(Q_i)$ for every i . Conversely, suppose that P admits a fixed-point-free isotone relation Φ with nonempty connected values. By Lemma 2.1,

$$\emptyset \neq \Phi(x) \subseteq \text{Inc}_P(x).$$

As $\Phi(x)$ is connected, there exists a component C_i with $\Phi(x) \subseteq C_i$. Define on Q_i

$$\Theta(q) = r_i[\Phi(q)] \quad (q \in Q_i).$$

Lemma 2.2 shows that Θ is isotone and has nonempty connected values. It remains to exclude fixed points. For $q \in Q_i \setminus \{x\}$, one has $r_i^{-1}(\{q\}) = \{q\}$, so a fixed point of Θ would already be a fixed point of Φ . The exceptional fiber is

$$r_i^{-1}(\{x\}) = \{x\} \cup (P \setminus Q_i).$$

However, the component selected above satisfies

$$\Theta(x) = r_i[\Phi(x)] = \Phi(x) \subseteq C_i,$$

which does not contain x . Therefore Θ is fixed-point-free, and Q_i fails CRFPP. This proves the converse implication in (1) by contraposition.

If $k \geq 2$, every Q_i omits another nonempty component, so it is proper. If $\text{Inc}_P(x)$ is empty, every nonempty value at x contains a point comparable with x ; by Lemma 2.1, no fixed-point-free isotone relation on P exists, giving $\text{CRFPP}(P)$. $\square$

Remark 3.2. No assumption of minimality, maximality, or retractability is imposed on x . When $k \geq 2$, the nonempty set $A_i = P \setminus Q_i = \bigcup_{j \neq i} C_j$ is retractable onto x in the classical sense of [3]. For $k = 1$, the retraction is the identity.

The new ingredient in Theorem 3.1 is not merely this retraction, but the equivalence (1), which depends on the connected value $\Phi(x)$ and the control of the exceptional fiber $r_i^{-1}(\{x\})$.

4. A RECURSIVE CHARACTERIZATION

The component-reduction theorem, together with the incomparability lemma, leads to a recursive description of the CRFPP. The two reduction operations are deletion of an irreducible point and localization to the components of an incomparability set.

Definition 4.1. Let $\mathcal{C}$ denote the *least isomorphism-invariant class* of finite nonempty posets satisfying the following rules:

- (i) Every one-point poset lies in $\mathcal{C}$.
- (ii) If a is irreducible in P and $P \setminus \{a\} \in \mathcal{C}$, then $P \in \mathcal{C}$.
- (iii) If, for some $x \in P$, the set $\text{Inc}_P(x)$ has at least two connected components $C_1, \dots, C_k$, and $N_P(x) \cup C_i \in \mathcal{C}$ for all i , then $P \in \mathcal{C}$.

Every poset occurring in a premise is strictly smaller than the poset in the corresponding conclusion. Hence this is a well-founded recursive definition entirely in terms of order structure.

Theorem 4.2 (Recursive characterization). *For every finite nonempty poset P ,*

$$\boxed{\text{CRFPP}(P) \iff P \in \mathcal{C}.}$$

Proof. Singletons have CRFPP. Lemma 2.3 and Theorem 3.1 show that rules (ii) and (iii), respectively, preserve this property. Induction on the generation of $\mathcal{C}$ gives $P \in \mathcal{C} \implies \text{CRFPP}(P)$.

For the reverse implication, induct on $|P|$. There is nothing to show for a singleton. Suppose $|P| > 1$ and $\text{CRFPP}(P)$. If P has an irreducible point a , the induced poset $P \setminus \{a\}$ has CRFPP by Lemma 2.3. It belongs to $\mathcal{C}$ by induction, so rule (ii) applies.

Otherwise, Lemma 2.4 supplies the fixed-point-free isotone relation $\Gamma_P(p) = \text{Inc}_P(p)$, with all values nonempty. Were all these values connected, it would contradict $\text{CRFPP}(P)$. Thus $\text{Inc}_P(x)$ is disconnected for some x , with at least two nonempty components. By Theorem 3.1, all the proper retracts $Q_i = N_P(x) \cup C_i$ have CRFPP; by induction they belong to $\mathcal{C}$. Rule (iii) now gives $P \in \mathcal{C}$. $\square$

5. FORBIDDEN RETRACTS AND EXPLICIT WITNESSES

The recursive characterization also admits a formulation in terms of forbidden retracts. The obstructions are precisely the nonsingleton posets to which neither of the preceding reduction operations applies.

Definition 5.1. Let $\mathcal{B}$ be the class of finite posets Q such that

$$\begin{aligned} |Q| &> 1, & Q \text{ has no irreducible points,} \\ \text{Inc}_Q(q) &\text{ is nonempty and connected for all } q \in Q. \end{aligned}$$

The definition of $\mathcal{B}$ involves only the order relation and connectivity; it does not quantify over multifunctions. For instance, the two-element antichain belongs to $\mathcal{B}$. We do not claim that $\mathcal{B}$ is minimal or irredundant as a class of obstructions.

Theorem 5.2 (Forbidden-retract characterization). *A finite nonempty poset P has the connected relational fixed point property if and only if no retract of P belongs to $\mathcal{B}$.*

Proof. If $Q \in \mathcal{B}$, the relation $\Gamma_Q(q) = \text{Inc}_Q(q)$ is nonempty-valued, isotone, connected-valued, and fixed-point-free by Lemma 2.4. Thus $\text{CRFPP}(Q)$ fails. Lemma 2.2 implies that no poset with CRFPP can have Q as a retract.

Conversely, suppose P lacks CRFPP and choose, among its retracts lacking CRFPP, one of minimum cardinality, say Q . Such a choice is possible because P is its own retract. The poset Q is not a singleton. If Q had an irreducible point, deleting it would produce, by Lemma 2.3, a smaller retract lacking CRFPP. If $\text{Inc}_Q(x)$ were disconnected, then Theorem 3.1 would produce a smaller component retract lacking CRFPP. In either case, composing the retractions yields a proper retract of the original P , contradicting the minimality of Q . Therefore Q has no irreducible points and no disconnected incomparability sets. By Lemma 2.4, all its incomparability sets are nonempty. Hence $Q \in \mathcal{B}$. $\square$

Corollary 5.3 (Explicit witnesses for failure). *If P lacks CRFPP, there are a retract $Q \in \mathcal{B}$ and an order-preserving retraction $r : P \rightarrow Q$ for which*

$$\Phi(p) = \text{Inc}_Q(r(p)) \subseteq Q \subseteq P$$

is a fixed-point-free isotone relation with nonempty connected values on P .

Proof. Theorem 5.2 supplies Q and r . The composition of the order-preserving retraction with the incomparability relation on Q is isotone and has nonempty connected values. If $p \in \Phi(p)$, then $p \in Q$, hence $r(p) = p$, which would give $p \in \text{Inc}_Q(p)$, an impossibility. $\square$

6. CONSEQUENCES AND AN EXAMPLE

6.1. A decision procedure and certificates. Theorem 4.2 gives a direct decision algorithm. On the current nonempty induced subposet R , accept if $|R| = 1$. Otherwise, if there is an irreducible point, delete it and recurse. If there is no such point but some x has disconnected $\text{Inc}_R(x)$, recurse on every component retract $N_R(x) \cup C$ and accept if and only if all branches accept. If neither reduction applies, reject. By Lemma 2.4, the final rejected state belongs to $\mathcal{B}$. Every reduction strictly decreases cardinality; any eligible choice of a point is valid by the equivalences proved above.

A positive decision can be recorded as a finite reduction tree, with singleton leaves. For a negative decision, it suffices to exhibit a retract $Q \in \mathcal{B}$ and an order-preserving retraction $r : P \rightarrow Q$. This pair has polynomial size and its defining properties can be checked in polynomial time. Corollary 5.3 then gives an explicit fixed-point-free connected-valued isotone relation on P . In particular, recognition of CRFPP belongs to coNP.

6.2. Invariance under finite-space homotopy. Each finite T_0 topological space may be regarded as a finite poset, and conversely. An irreducible point of a finite poset is a *beat point* of the corresponding finite space. By Stong's core theorem [6], two finite T_0 spaces are homotopy equivalent if and only if their cores, obtained by successively deleting beat points, are homeomorphic.

Corollary 6.1. *The connected relational fixed point property is invariant under homotopy equivalence of finite nonempty T_0 spaces.*

Proof. Lemma 2.3 shows that deleting any beat point preserves CRFPP in both directions. Hence a finite T_0 space has CRFPP if and only if its core does. The result follows from Stong's characterization of homotopy equivalence in terms of isomorphic cores [6, Theorems 2–4, pp. 329–330]. $\square$

6.3. An example without irreducible points. The component rule can apply even when no single irreducible point can be removed. The following small example also shows how a connected-valued fixed point property can hold without I -dismantlability to a singleton.

Example 6.2. Let H be the fourteen-element poset with minima a, b, c, d, e , intermediate elements u, v, w, x, y, z , and maxima r, s, t . Its covering relations are determined by the table:

	lower covers	upper covers
u	a, b	r, s
v	c, d	r, s
w	a, e	r, s
x	b, e	r, t
y	c, e	r, s
z	d, e	s, t

There are no other covers. Each element with covers in either direction has at least two of them, so H has no irreducible points. However,

$$\text{Inc}_H(e) = \{a, b, u\} \dot{\cup} \{c, d, v\},$$

where both displayed sets are connected components.

By Theorem 3.1, it suffices to verify CRFPP of the two eleven-element component retracts. In the retract retaining $\{a, b, u\}$, the following sequence consists of successive deletions of irreducible points:

$$y, z, t, x, b, e, u, a, r, w;$$

only s remains. The other retract is isomorphic to this one under the order automorphism

$$(a\ c)(b\ d)(u\ v)(w\ y)(x\ z)(r\ s)$$

fixing e, t . Thus both retracts have CRFPP by Lemma 2.3, and hence so does H .

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