

RANKWISE AUTOMORPHISM EXTENSION IN FINITE POSETS

PATRIZIO CINTIOLI

ABSTRACT. Let P be a finite poset in which any two points of the same rank have isomorphic one-point-deletion cards. We prove that every isomorphism between such cards extends uniquely, by mapping the deleted points to one another, to an automorphism of P . In particular, the automorphism group of P acts transitively on each rank level, and the converse also holds. No primeness or indecomposability assumption is needed. The proof first shows, by counting longest chains, that every point lies on a longest chain. It then recovers original ranks within deletion cards, derives biregularity between rank levels, and detects deleted comparabilities through degree deficits. As an application, we confirm Conjecture 6.2 of Ille and Rampon, characterizing prime k -morphic posets of height $k - 1$ by rankwise transitivity.

1. INTRODUCTION AND STATEMENTS

All ambient posets P considered here are finite and nonempty, but deletion cards $P - v$ may be empty. For $v \in P$, let $P - v$ denote the induced subposet on $P \setminus \{v\}$. We write $\rho_P(v)$ for the rank of v , and $L_r(P) = \{v \in P : \rho_P(v) = r\}$ for the rank- r level. The *card* of v is the isomorphism type of $P - v$.

A basic distinction in reconstruction theory is between two points having isomorphic cards and two points being equivalent under an automorphism of P . In general, isomorphic cards need not arise from automorphic points. This note shows that these notions coincide *within rank levels* when all points on each level have isomorphic cards. Moreover, every isomorphism between the corresponding cards extends.

Theorem 1.1 (Rankwise extension and transitivity). *Let P be a finite nonempty poset. The following conditions are equivalent:*

- (i) *whenever $\rho_P(v) = \rho_P(w)$, one has $P - v \cong P - w$;*
- (ii) *$\text{Aut}(P)$ acts transitively on $L_r(P)$ for each rank r of P .*

Moreover, if these conditions hold and v, w have equal rank, then every isomorphism $f : P - v \rightarrow P - w$ extends to an automorphism of P by putting $\hat{f}(v) = w$.

The nontrivial implication and the extension property are proved in Section 5. The main assumption, used throughout, is

$$(H2) \quad \rho_P(v) = \rho_P(w) \implies P - v \cong P - w.$$

It automatically implies the full-height identity

$$(H1) \quad \rho_P(x) + \bar{\rho}_P(x) = \text{ht}(P) \quad (x \in P),$$

where $\bar{\rho}_P$ denotes corank. This implication, not needed as a separate hypothesis, follows from a count of longest chains in Lemma 2.3.

Let $\mathcal{S}(P) = \{[P - v] : v \in P\}$ be the set of isomorphism types of one-point-deletion cards; multiplicities are *not* recorded. Following Ille and Rampon [1], P is called *k -morphic* when $|\mathcal{S}(P)| = k$. In their Theorem 1.16, Ille and Rampon characterized prime bimorphic posets and subsequently formulated Conjecture 6.2, concerning prime k -morphic posets of height

2020 *Mathematics Subject Classification.* 06A06, 05C60.

Key words and phrases. Finite partially ordered set, deletion card, automorphism, rank, corank, k -morphic poset, reconstruction.

$k - 1$ for arbitrary $k \geq 2$. We settle this conjecture in the affirmative, as an application of the rankwise extension theorem together with their published results.

Theorem 1.2 (Ille–Rampon Conjecture 6.2). *Let $k \geq 2$ and let P be a finite prime poset of height $k - 1$. Then P is k -morphic if and only if $\text{Aut}(P)$ is transitive on each level $L_r(P)$, $0 \leq r \leq k - 1$.*

The proof of the rankwise extension theorem does not use primeness or any external reconstruction result. The specialization to prime posets uses Fact 6.1 and Theorems 3.1–3.2 of Ille and Rampon [1], the latter two based on work of Schröder [2].

2. PRELIMINARIES: RANKS, CORANKS AND VERTEX DELETION

In this section we fix the notation and recall the elementary facts about ranks, coranks, and vertex deletion needed in the sequel. For general background on partially ordered sets, we refer the reader to Schröder [3].

Definition 2.1. For a nonempty finite poset P , its height $h = \text{ht}(P)$ is $|C| - 1$ for a longest chain C in P . For $x \in P$, define

$$\begin{aligned}\rho_P(x) &= \max\{|C| - 1 : C \subseteq P \text{ a chain with greatest element } x\}, \\ \bar{\rho}_P(x) &= \max\{|C| - 1 : C \subseteq P \text{ a chain with least element } x\}.\end{aligned}$$

Thus ρ_P is the rank and $\bar{\rho}_P$ is the corank (or dual rank). Set

$$L_i(P) = \{x \in P : \rho_P(x) = i\} \quad (0 \leq i \leq h).$$

For $x \in P$, write

$$d_P(x) = |\{y \in P : y < x\}|, \quad u_P(x) = |\{y \in P : x < y\}|.$$

These are the *strict* lower and upper degrees; they count all strict comparabilities, not just covering relations. For $v \in P$, the card $P - v$ is the induced order on $P \setminus \{v\}$.

We begin by recalling some elementary properties of ranks and coranks. In particular, these properties describe how longest chains intersect the rank levels of a finite poset.

Lemma 2.2 (Elementary rank properties). *Let P be a nonempty finite poset of height h .*

- (a) *If $x < y$ then $\rho_P(x) < \rho_P(y)$ and $\bar{\rho}_P(x) > \bar{\rho}_P(y)$. Consequently each $L_i(P)$ is an antichain.*
- (b) *The rank levels $L_0(P), \dots, L_h(P)$ are all nonempty. If $x_0 < \dots < x_h$ is a chain of length h , then $\rho_P(x_i) = i$ and $\bar{\rho}_P(x_i) = h - i$.*
- (c) *For every $x \in P$ one has $\rho_P(x) + \bar{\rho}_P(x) \leq h$. Equality holds if and only if x lies on at least one chain of length h .*

Proof. (a) Take a longest chain whose greatest point is x and append y . This extends it by one point, so $\rho_P(y) \geq \rho_P(x) + 1$. Applying the dual argument to a longest chain starting at y gives $\bar{\rho}_P(x) \geq \bar{\rho}_P(y) + 1$. Thus two distinct points on the same level cannot be comparable.

(b) For each i , the initial segment $x_0 < \dots < x_i$ proves $\rho_P(x_i) \geq i$. The final segment $x_i < \dots < x_h$ proves $\bar{\rho}_P(x_i) \geq h - i$. If either inequality were strict, concatenating a longer appropriate chain with the other segment would create a chain longer than h , a contradiction. Thus both are equalities. Every $L_i(P)$ contains x_i .

(c) Take a longest chain ending at x , of length $\rho_P(x)$, and a longest chain starting at x , of length $\bar{\rho}_P(x)$. The two chains intersect only at x : all points of the first, except x , are strictly below x , whereas all points of the second, except x , are strictly above x . Their union, read in the indicated order, is a chain of length $\rho_P(x) + \bar{\rho}_P(x)$. This length is at most h . If equality holds, the union is a longest chain through x . Conversely, if a longest chain passes through x , its lower and upper segments show that $\rho_P(x) + \bar{\rho}_P(x) \geq h$; the reverse inequality was just proved. $\square$

The following counting argument is the first key step towards the extension theorem. It shows that the full-height identity (H1) is already a consequence of the rankwise card-isomorphism hypothesis (H2).

Lemma 2.3 (Longest chains and automatic full height). *Suppose P satisfies (H2) and put $h = \text{ht}(P)$. Let $N_h(P)$ denote the number of chains with $h + 1$ points, and let $N_h(P; v)$ be the number of such chains containing v . Then for every $r \in \{0, \dots, h\}$ and every $v \in L_r(P)$,*

$$(1) \quad N_h(P; v) = \frac{N_h(P)}{|L_r(P)|} > 0.$$

In particular, every point of P belongs to a longest chain, and the full-height identity (H1) holds.

Proof. Every chain of $P - v$ is a chain of P . Therefore

$$N_h(P - v) = N_h(P) - N_h(P; v).$$

For fixed r , hypothesis (H2) makes $N_h(P - v)$ independent of $v \in L_r(P)$, so $N_h(P; v)$ is a constant c_r on that level. By Lemma 2.2(b), every longest chain has exactly one point on each level $L_r(P)$. Counting pairs consisting of a longest chain and its point in $L_r(P)$ gives

$$|L_r(P)|c_r = N_h(P).$$

Since P is nonempty, $N_h(P) \geq 1$; hence $c_r > 0$. Thus every $v \in P$ lies on a longest chain, and Lemma 2.2(c) yields (H1). The argument also covers $h = 0$, when longest chains are singletons. $\square$

We also need a bound on how much ranks and coranks can change under a one-point deletion. Unlike the preceding lemma, this bound holds for arbitrary finite posets.

Lemma 2.4 (A one-point deletion changes ranks by at most one). *Let $v, x \in P$ be distinct and set $X = P - v$. Then*

$$(2) \quad \rho_P(x) - 1 \leq \rho_X(x) \leq \rho_P(x),$$

$$(3) \quad \bar{\rho}_P(x) - 1 \leq \bar{\rho}_X(x) \leq \bar{\rho}_P(x).$$

In particular, the corresponding differences are either zero or one.

Proof. Every chain of X is a chain of P , proving both upper bounds. Choose a longest chain of P ending at x , with $\rho_P(x) + 1$ points. If it does not contain v , it remains a chain of the same length in X . If it contains v , removing that point leaves an induced chain ending at x with one fewer point: transitivity of the order ensures that all remaining comparisons persist. Thus $\rho_X(x) \geq \rho_P(x) - 1$. The analogous argument with a longest chain starting at x proves the corank inequality. These arguments include the rank-zero and corank-zero cases, when only the value zero is possible. $\square$

3. RECOVERING THE ORIGINAL RANK INSIDE A CARD

The rank of a surviving vertex can decrease after a deletion. Consequently, one cannot simply assume that an isomorphism of cards preserves the *original* ranks. The following formula resolves exactly this point.

Lemma 3.1 (Rank recovery from a specified deletion level). *Let P satisfy (H1), let $v \in L_r(P)$, and write $X = P - v$. For $x \in X$, define*

$$(4) \quad \ell_r^X(x) = \begin{cases} \rho_X(x), & \rho_X(x) < r, \\ h - \bar{\rho}_X(x), & \rho_X(x) \geq r. \end{cases}$$

Then

$$\boxed{\ell_r^X(x) = \rho_P(x) \quad (x \in X).}$$

The formula applies when $r = 0$ or $r = h$, and it makes no assumption that any two cards of P are isomorphic.

Proof. Fix $x \in X$ and put $i = \rho_P(x)$. We distinguish three mutually exclusive and exhaustive cases.

Case 1: $i < r$. If $v < x$, the strict rank inequality of Lemma 2.2(a) would give $r = \rho_P(v) < \rho_P(x) = i$, contradicting $i < r$. Thus v is not in the principal down-set of x . All chains ending at x are unchanged by deletion of v ; hence $\rho_X(x) = \rho_P(x) = i < r$. The first branch of (4) returns i .

Case 2: $i = r$. Since two different elements of the same rank are incomparable, neither $v < x$ nor $x < v$. The principal down-set and up-set of x are both unchanged. Therefore

$$\rho_X(x) = \rho_P(x) = r, \quad \bar{\rho}_X(x) = \bar{\rho}_P(x) = h - r,$$

where the last equality uses (H1). The second branch of (4), which is selected because $\rho_X(x) = r$, gives $h - (h - r) = r = i$.

Case 3: $i > r$. If $x < v$, Lemma 2.2(a) would imply $i = \rho_P(x) < \rho_P(v) = r$, an impossibility. Thus the principal up-set of x is unchanged, giving

$$\bar{\rho}_X(x) = \bar{\rho}_P(x) = h - i$$

by (H1). By Lemma 2.4,

$$\rho_X(x) \geq \rho_P(x) - 1 = i - 1 \geq r.$$

Hence the second branch is selected, and it equals $h - (h - i) = i$.

The three cases yield the formula for every surviving x . Notice that the ambient height h and the rank r of the deleted vertex are *parameters* in (4); the lemma does not claim to infer r from an unlabeled card. $\square$

Corollary 3.2 (Original ranks are preserved by equal-rank card isomorphisms). *Assume (H1). Let $v, w \in L_r(P)$ and let $f : P - v \rightarrow P - w$ be an isomorphism. Then*

$$\rho_P(f(x)) = \rho_P(x) \quad (x \in P \setminus \{v\}).$$

Proof. Order isomorphisms send chains bijectively onto chains and preserve the greatest and least elements of chains. Consequently, f preserves both rank and corank *computed in the two cards*:

$$\rho_{P-w}(f(x)) = \rho_{P-v}(x), \quad \bar{\rho}_{P-w}(f(x)) = \bar{\rho}_{P-v}(x).$$

Apply the same expression (4), with the same parameters h and r , to the two cards. Its value at $f(x)$ equals its value at x . By Lemma 3.1 these values are respectively $\rho_P(f(x))$ and $\rho_P(x)$. $\square$

4. UNIFORM INCIDENCES BETWEEN RANK LEVELS

Write $L_i = L_i(P)$ and let

$$E_{ij}(P) = |\{(x, y) \in L_i \times L_j : x < y\}| \quad (0 \leq i < j \leq h).$$

If $X = P - v$, where v has rank r , the corresponding observable in X must use the recovered original levels:

$$(5) \quad E_{ij}^{(r)}(X) = |\{(x, y) \in X \times X : \ell_r^X(x) = i, \ell_r^X(y) = j, x <_X y\}|.$$

With h, r, i, j fixed, this is an isomorphism invariant of X , by the rank recovery lemma and its corollary.

Lemma 4.1 (Incidence accounting for a single deletion). *Assume (H1) and fix $0 \leq i < j \leq h$. Then*

$$(6) \quad E_{ij}^{(i)}(P - v) = E_{ij}(P) - |\{y \in L_j : v < y\}| \quad (v \in L_i),$$

$$(7) \quad E_{ij}^{(j)}(P - w) = E_{ij}(P) - |\{x \in L_i : x < w\}| \quad (w \in L_j).$$

Proof. By Lemma 3.1, the vertices in the first coordinate of (5) are exactly the surviving elements of the original level L_i and those in the second coordinate are exactly the surviving elements of L_j . Since the order on the surviving vertices is induced, every comparison between two surviving vertices remains unchanged.

When $v \in L_i$ is deleted, the only comparisons counted by $E_{ij}(P)$ that disappear are the comparisons (v, y) with $y \in L_j$ and $v < y$. Their number is the term subtracted in (6). The argument for $w \in L_j$ is identical, with the deleted vertex in the second coordinate. No comparison between two surviving vertices is lost, and no new one is created. $\square$

Lemma 4.2 (Rankwise biregularity). *Suppose (H1) and (H2) hold. For each pair $0 \leq i < j \leq h$, there are nonnegative integers a_{ij} and b_{ij} such that*

$$(8) \quad |\{y \in L_j : x < y\}| = a_{ij} \quad \text{for all } x \in L_i,$$

$$(9) \quad |\{x \in L_i : x < y\}| = b_{ij} \quad \text{for all } y \in L_j.$$

Moreover,

$$(10) \quad |L_i|a_{ij} = |L_j|b_{ij} = E_{ij}(P).$$

In particular, the total numbers of strict predecessors and strict successors are constant on each rank level:

$$(11) \quad d_P(x) = D_i := \sum_{j < i} b_{ji}, \quad u_P(x) = U_i := \sum_{j > i} a_{ij} \quad \text{whenever } x \in L_i.$$

Empty sums are zero.

Proof. Fix $i < j$. For any $v, w \in L_i$, hypothesis (H2) gives $P - v \cong P - w$. Since $E_{ij}^{(i)}$ is an isomorphism invariant of a card for which the deleted rank is i ,

$$E_{ij}^{(i)}(P - v) = E_{ij}^{(i)}(P - w).$$

Using (6) on both sides and canceling $E_{ij}(P)$ yields

$$|\{y \in L_j : v < y\}| = |\{y \in L_j : w < y\}|.$$

Thus the number of successors in L_j is independent of the choice of $v \in L_i$; call its value a_{ij} .

Likewise, for any $v, w \in L_j$, use (H2) and (7). Canceling $E_{ij}(P)$ now gives

$$|\{x \in L_i : x < v\}| = |\{x \in L_i : x < w\}|.$$

This common value is b_{ij} . This establishes both uniformity assertions, with no transitivity assumption on $\text{Aut}(P)$.

Every pair counted in $E_{ij}(P)$ has exactly one lower endpoint in L_i and one upper endpoint in L_j . Counting first by its lower endpoint gives $E_{ij}(P) = |L_i|a_{ij}$; counting by the upper endpoint gives $E_{ij}(P) = |L_j|b_{ij}$.

Finally, Lemma 2.2(a) implies that every strict predecessor of $x \in L_i$ lies in exactly one level L_j with $j < i$, and every strict successor lies in exactly one level L_j with $j > i$. Adding the respective uniform incidence numbers over these disjoint levels proves (11). $\square$

5. RECOVERING THE DELETED VERTEX AND EXTENDING THE ISOMORPHISM

Using the recovery of original ranks and the uniformity of degree counts established in the preceding sections, we now show how to reconstruct all comparabilities involving a deleted vertex. This will yield the extension theorem for isomorphisms between cards of vertices of the same rank.

Lemma 5.1 (The degree-deficit criteria). *Assume (H1) and (H2). Let $v \in L_r$ and $X = P - v$. For each surviving x let $i = \ell_r^X(x) = \rho_P(x)$. Then:*

(a) *if $i < r$, necessarily $v \not< x$, and*

$$(12) \quad x < v \iff u_X(x) = U_i - 1;$$

(b) if $i > r$, necessarily $x \not\leq v$, and

$$(13) \quad v < x \iff d_X(x) = D_i - 1;$$

(c) if $i = r$, then x and v are incomparable.

Thus all strict relations involving v can be recovered from X , provided h , r and the rankwise totals D_i, U_i are specified. No assertion is made that these parameters are recoverable from an unlabeled set of cards alone.

Proof. If $i < r$, strict increase of ranks excludes $v < x$. Removing v decreases the number of strict successors of x by exactly one when $x < v$, and by zero when x is incomparable with v . Thus

$$u_X(x) = u_P(x) - \mathbf{1}_{\{x < v\}} = U_i - \mathbf{1}_{\{x < v\}}.$$

The indicator has value zero or one, so equality with $U_i - 1$ is equivalent to $x < v$. This proves (a), even when $U_i = 0$: in that case the equality $u_X(x) = U_i - 1$ cannot occur.

If $i > r$, strict increase of ranks excludes $x < v$. Similarly, the number of predecessors of x decreases by exactly one when $v < x$, giving

$$d_X(x) = d_P(x) - \mathbf{1}_{\{v < x\}} = D_i - \mathbf{1}_{\{v < x\}}.$$

This proves (b), with the analogous boundary convention when $D_i = 0$.

If $i = r$, distinct elements of the same rank are incomparable by Lemma 2.2(a). This proves (c). $\square$

Theorem 5.2 (Full rankwise extension theorem). *Let P be a finite nonempty poset of height h satisfying (H2). Fix $r \in \{0, \dots, h\}$, $v, w \in L_r$ and any isomorphism $f : P - v \rightarrow P - w$. Then its extension $\hat{f}$ by $v \mapsto w$ is an automorphism of P . Consequently $\text{Aut}(P)$ acts transitively on every rank level. Conversely, rankwise transitivity implies (H2).*

Proof. Lemma 2.3 first gives (H1) from the sole hypothesis (H2). Thus all the rank-recovery and biregularity lemmas apply. By Corollary 3.2, for each $x \in P \setminus \{v\}$ we have

$$i := \rho_P(x) = \rho_P(f(x)).$$

The card isomorphism f also preserves the strict numbers of predecessors and successors within the cards:

$$(14) \quad u_{P-v}(x) = u_{P-w}(f(x)), \quad d_{P-v}(x) = d_{P-w}(f(x)).$$

Let $x \neq v$. If $i < r$, by Lemma 5.1(a),

$$\begin{aligned} x < v &\iff u_{P-v}(x) = U_i - 1 \\ &\iff u_{P-w}(f(x)) = U_i - 1 \\ &\iff f(x) < w. \end{aligned}$$

The middle equivalence follows from (14). The same constant U_i applies at $f(x)$ because both original ranks are i . This handles all possible comparabilities involving v and a vertex of lower rank.

If $i > r$, Lemma 5.1(b) similarly gives

$$\begin{aligned} v < x &\iff d_{P-v}(x) = D_i - 1 \\ &\iff d_{P-w}(f(x)) = D_i - 1 \\ &\iff w < f(x). \end{aligned}$$

If $i = r$, Lemma 5.1(c) says both $x \parallel v$ and $f(x) \parallel w$. There are no additional possible cross-level orientations, since strict comparability forces strictly increasing ranks.

Define $\hat{f}|_{P-v} = f$ and $\hat{f}(v) = w$. Because f bijects $P \setminus \{v\}$ with $P \setminus \{w\}$, this extension is a bijection of P . If two points are different from v , their relation is preserved in both directions because f is an isomorphism of the induced cards. If one point equals v , the two

chains of equivalences above, or the equal-rank incomparability, show that $\hat{f}$ preserves and reflects its relation with the other point. Reflexivity is automatic. Thus

$$x \leq_P y \iff \hat{f}(x) \leq_P \hat{f}(y) \quad \text{for all } x, y \in P.$$

Hence $\hat{f}$ is an automorphism.

Finally, for any $v, w \in L_r$, hypothesis (H2) provides at least one isomorphism $P - v \rightarrow P - w$; its extension sends v to w . Therefore $\text{Aut}(P)$ is transitive on L_r . Conversely, an automorphism taking $v \in L_r$ to $w \in L_r$ restricts to an isomorphism from $P - v$ onto $P - w$. This completes the equivalence and covers $|L_r| = 1$ and $h = 0$. $\square$

Remark 5.3 (Strength and limitations). The proof uses the full-height property (H1) to recover the *original* ranks in a card, but Lemma 2.3 shows that this property already follows from rankwise card isomorphism (H2). The latter also gives uniform incidences and the degree-deficit rules. The proof does not use primeness, modular decomposition or equality of full decks. The rank r of the deleted point is a parameter, so the theorem does not assert reconstruction from the unlabeled set $\mathcal{S}(P)$.

6. ILLE–RAMPON’S CONJECTURE 6.2

We recall the precise formulation of Conjecture 6.2 of Ille and Rampon [1, p. 194]. Following their terminology, a poset P is *W-transitive*, for $W \subseteq P$, if for every $v, w \in W$ there exists an automorphism φ of P such that $\varphi(v) = w$.

Conjecture 6.2 (Ille–Rampon). Let $k \geq 2$, and let P be a prime partial order such that $\text{ht}(P) = k - 1$. Then P is k -morphic if and only if the following condition holds:

$$(U1) \quad \begin{array}{l} \text{for every } 0 \leq \ell \leq k - 1, \\ P \text{ is } \{v \in P : \rho_P(v) = \ell\}\text{-transitive.} \end{array}$$

We prove this conjecture using the general rankwise extension theorem together with Fact 6.1 and Theorems 3.1–3.2 of Ille and Rampon, recalled below.

Definition 6.1. A subset M of a poset P is a *module* if, for each $x \in P \setminus M$, one of the following relations holds uniformly for all $m \in M$: $x < m$, $m < x$, or $x \parallel m$. A module is *nontrivial* if $2 \leq |M| < |P|$. A finite poset is *prime* if it has at least three elements and has no nontrivial modules. Write $v \sim_P w$ if $P - v \cong P - w$.

Remark 6.2 (Published results used in the proof). The proof of Conjecture 6.2 relies on the following results of Ille and Rampon.

- (E1) **Ille–Rampon, Fact 6.1** [1, pp. 193–194]. For a prime k -morphic poset P of height $k - 1$, with $k \geq 2$, the published statement asserts: (T1) each rank level is exactly one $\sim_P$ -class; (T2) each corank level is exactly one $\sim_P$ -class; (T3) $\rho_P(x) + \bar{\rho}_P(x) = k - 1$ for every x . *Only (T1) is needed* below: it implies (H2), which already entails (T3) by Lemma 2.3.
- (E2) **Ille–Rampon, Theorem 3.1** [1, p. 187], **from Schröder** [2]. If a connected poset has two vertices v, w with $v \sim_P w$ and $\rho_P(v) < \rho_P(w)$, then *unless* v is minimal and w is maximal, P has a nontrivial module. The published theorem actually gives stronger information about this module, which is not needed here.
- (E3) **Ille–Rampon, Theorem 3.2** [1, p. 187], **from Schröder** [2]. If a connected poset is not a chain and a minimal vertex v and maximal vertex w satisfy $v \sim_P w$, then their $\sim_P$ -class contains no third vertex.

These results are used in the forms stated above. The remaining arguments required for the proof are provided below.

We first establish some elementary structural properties of finite prime posets. These will ensure that the published results recalled above apply in the proof of Conjecture 6.2.

Lemma 6.3 (Structural elementary facts about prime posets). *Every finite prime poset P is connected in its comparability graph, is not a chain, and has at least two minimal and at least two maximal vertices. It has height at least one.*

Proof. Since P is prime, $|P| \geq 3$. Suppose its comparability graph is disconnected. If there is a component C with $2 \leq |C| < |P|$, then C is a nontrivial module: every vertex outside it is incomparable with all points of C . If every component is a singleton, then any pair of distinct vertices is a nontrivial module, because $|P| \geq 3$. Both possibilities contradict primeness. Thus the comparability graph is connected.

If P is a chain with at least three points, the first two points of that chain form a nontrivial module: all remaining points lie strictly above both. Hence a prime poset cannot be a chain.

Suppose P has exactly one minimal point m . For any other $y \in P$, finiteness ensures that a minimal element lies below y (descend along strict inequalities until no further descent is possible). It must be m , so $m < y$. Consequently $P \setminus \{m\}$ is a module, since the only outside vertex m lies below every element of it. Its cardinality is $|P| - 1 \geq 2$, making it nontrivial. This is impossible. Thus P has at least two minima. The proof for maximal vertices is dual.

Finally, if the height were zero, P would be an antichain. Its comparability graph would then be disconnected, a contradiction. Therefore $h \geq 1$. $\square$

The key step in the converse direction of Conjecture 6.2 is to show that vertices of different ranks cannot have isomorphic deletion cards under rankwise transitivity. We establish this using the preceding lemma and the published results recalled above.

Lemma 6.4 (No cross-rank equivalence under level transitivity). *Let P be a finite prime poset such that $\text{Aut}(P)$ is transitive on each rank level. If $\rho_P(v) \neq \rho_P(w)$, then $P - v \not\cong P - w$.*

Proof. Suppose otherwise. Interchange v, w if necessary to arrange $\rho_P(v) < \rho_P(w)$. By Lemma 6.3, P is connected and prime. By Theorem 3.1 of Ille and Rampon, recalled in (E2), v must be minimal and w maximal: otherwise P would have a nontrivial module, contradicting primeness.

By Lemma 6.3 choose a second minimal vertex $v' \neq v$. Since the minimal vertices are exactly the rank-zero level, level transitivity supplies an automorphism g of P with $g(v) = v'$. Restricting g to the complement of v gives an isomorphism $P - v \cong P - v'$. Combining this with the assumed isomorphism $P - v \cong P - w$, we obtain $v' \sim_P v \sim_P w$. The three vertices are distinct: $v' \neq v$ by construction and $v' \neq w$ because w is maximal, while v' is minimal, and the prime poset has height at least one with no isolated vertex.

But P is connected and not a chain by Lemma 6.3. By Theorem 3.2 of Ille and Rampon, recalled in (E3), a minimal–maximal pair of vertices with isomorphic cards forms a complete $\sim_P$ -class consisting of exactly those two vertices. This excludes v' and is a contradiction. Hence different ranks cannot produce isomorphic cards. $\square$

Theorem 6.5 (Resolution of Conjecture 6.2 of Ille and Rampon). *Let $k \geq 2$ and let P be a finite prime poset with $\text{ht}(P) = k - 1$. Then P is k -morphic if and only if $\text{Aut}(P)$ acts transitively on each rank level.*

Proof. Forward implication. Suppose P is k -morphic. By Fact 6.1 of Ille and Rampon, recalled in (E1), each rank level is a $\sim_P$ -class; in particular,

$$\rho_P(v) = \rho_P(w) \implies P - v \cong P - w.$$

Thus condition (H2) holds, and Theorem 5.2 applies directly. In particular, Lemma 2.3 also supplies the full-height identity without a separate appeal to part (T3) of Fact 6.1. Every isomorphism between two cards obtained by deleting vertices on the same rank level extends to an automorphism mapping the deleted vertices to each other. In particular, the vertices of each rank level form an orbit of $\text{Aut}(P)$.

Reverse implication. Suppose $\text{Aut}(P)$ acts transitively on every rank level. If $v, w \in L_i(P)$, choose an automorphism taking v to w and restrict it to $P - v$. Hence $P - v \cong P - w$: all vertices of a given rank produce the same card type. By Lemma 6.4, card types corresponding to *different* ranks are distinct.

By Lemma 2.2(b), all $h + 1 = k$ rank levels $L_0(P), \dots, L_h(P)$ are nonempty. There is therefore exactly one card type for each of these k nonempty levels, and distinct levels give distinct types. Thus

$$|\mathcal{S}(P)| = h + 1 = k.$$

This proves the converse and the theorem. $\square$

Remark 6.6 (A shorter derivation of Fact 6.1(T3)). The longest-chain counting argument shows that (T1) of Fact 6.1 already implies the full-height identity (T3): each rank level being a $\sim_P$ -class implies (H2). Thus (T2) is unnecessary for the deduction of (T3), although both are useful structural assertions in the original paper.

7. CONSEQUENCES AND LIMITATIONS

We conclude with a consequence of the extension theorem for automorphism groups, followed by a clarification of its scope in relation to reconstruction.

Corollary 7.1 (Stabilizers and card automorphisms). *Let P satisfy (H2), let r be a rank and let $v \in L_r(P)$. Restriction induces a group isomorphism*

$$\text{Aut}(P)_v \cong \text{Aut}(P - v),$$

where $\text{Aut}(P)_v$ is the point stabilizer of v . Consequently,

$$(15) \quad |\text{Aut}(P)| = |L_r(P)| |\text{Aut}(P - v)|.$$

Proof. An automorphism fixing v restricts to an automorphism of $P - v$. This restriction map is injective, since it fixes every point if its restriction is the identity. It is surjective by Theorem 5.2, applied with $w = v$ to any automorphism of $P - v$. Rankwise transitivity implies that the orbit of v is the entire level $L_r(P)$, so (15) follows from the orbit-stabilizer formula. $\square$

Remark 7.2. The theorem concerns isomorphisms between deletion cards of points of the *same* rank. It does not claim that isomorphic cards of points of different ranks imply those points are automorphic: in a nontrivial chain all deletion cards are isomorphic, whereas every rank level is a singleton. Nor does the theorem by itself resolve the problem of reconstructing P from the set $\mathcal{S}(P)$ without multiplicities.

8. CONCLUSIONS

We have shown that, for finite posets, the condition that vertices of the same rank have isomorphic one-point-deletion cards is equivalent to transitivity of the automorphism group on each rank level. More strongly, every isomorphism between such cards extends uniquely to an automorphism of the original poset. This result requires no primeness, connectivity, or indecomposability assumption.

As an application, we have resolved Conjecture 6.2 of Ille and Rampon, characterizing prime k -morphic posets of height $k - 1$ in terms of rankwise automorphism transitivity. Although our results do not settle the general set reconstruction problem for finite posets, they establish a precise connection between deletion-card isomorphisms and global automorphism structure.

ACKNOWLEDGMENTS

The main results of this paper were discovered by ChatGPT 6 Pro (OpenAI), which was also used in developing the proofs and preparing the manuscript. The author takes full responsibility for the correctness of the mathematical statements and proofs, for the accuracy of the citations and attributions, and for the final content of the manuscript.

REFERENCES

- [1] P. Ille and J.-X. Rampon, *Monomorphic and bimorphic partial orders*, Australasian Journal of Combinatorics **89**(2) (2024), 179–195. https://ajc.maths.uq.edu.au/pdf/89/ajc_v89_p179.pdf.
- [2] B. S. W. Schröder, *Pseudo-similar points in ordered sets*, Discrete Mathematics **310** (2010), 2815–2823.
- [3] B. S. W. Schröder, *Ordered Sets: An Introduction with Connections from Combinatorics to Topology*, 2nd ed., Birkhäuser, 2016.

MATHEMATICS DIVISION, SCHOOL OF SCIENCE AND TECHNOLOGY, UNIVERSITY OF CAMERINO, 62032 CAMERINO, ITALY

Email address: `patrizio.cintioli@unicam.it`