

RCA₀ PROVES THE COLLAPSE OF THE INTERSECTION PRINCIPLES HIERARCHY AT LEVEL TWO

PATRIZIO CINTIOLI

ABSTRACT. This paper shows that the pairwise intersection principle (2IP) implies, over RCA₀, the existence of a Cohen 1-generic real relative to every given set. Together with the known implications, this yields the equivalence over RCA₀ of the finite intersection principle (FIP), 2IP, and 1-GEN. In fact, FIP is equivalent to n IP for every fixed $n \geq 2$. These results give affirmative answers to Questions 1 and 2 of Cholak, Downey, and Igusa, and to Question 9.10.26 of Dzhafarov and Mummert. The proof uses cumulative, nested supports satisfying a finite Helly property and a decoding argument based on a least selected label. Neither Σ_2^0 induction nor Σ_2^0 bounding is required.

1. INTRODUCTION AND CONVENTIONS

The *finite intersection principle* (FIP), studied in reverse mathematics by Dzhafarov and Mummert [4], asserts that every nontrivial countable family of subsets of $\mathbb{N}$ has a maximal subfamily in which every finite collection of at least two distinct members has nonempty intersection.

For each fixed $n \geq 2$, the *n-intersection principle* (n IP) requires a maximal subfamily in which every collection of exactly n distinct members has nonempty intersection. In particular, 2IP is the pairwise intersection principle. These are the conventions of the original paper of Dzhafarov and Mummert [4], where distinctness is understood extensionally. The formulation in [5, Definition 9.10.17] instead uses distinct indices and, for FIP, also requires nonempty singleton intersections.

The family constructed below consists entirely of nonempty, pairwise distinct sets. Thus, for the constructed instance, the two formulations of FIP and 2IP agree. The known implications involving higher intersection principles are used with the conventions of their cited sources.

Earlier degree-theoretic work on FIP and its variants includes Diamondstone, Downey, Greenberg, and Turetsky [3]. Cholak, Downey, and Igusa [1, Theorem 5.8] proved that FIP is equivalent, over RCA₀, to the principle 1-GEN asserting the existence of a Cohen 1-generic real relative to each given set. Their proof that 2IP implies 1-GEN, however, was formalized over the stronger system RCA₀ + $I\Sigma_2^0$; see [1, Proposition 5.7]. They explicitly asked in [1, Question 1] whether RCA₀ proves $\text{FIP} \leftrightarrow 2\text{IP}$. Their Question 2 asks whether there exist distinct integers $m, n \geq 2$ such that RCA₀ proves $m\text{IP} \leftrightarrow n\text{IP}$, and raises the same question over RCA₀ + $B\Sigma_2^0$. The closely related question whether RCA₀ + $2\text{IP} \rightarrow 1\text{-GEN}$ was later listed by Dzhafarov and Mummert in [5, Question 9.10.26].

In a different, higher-order setting, Sanders [6, Remark 3.9] showed that the uniform versions of FIP and of n IP, for $n \geq 2$, are equivalent to Feferman's search functional μ^2 . These uniform equivalences do not settle the corresponding nonuniform equivalences over RCA₀.

The main result of this paper establishes $2\text{IP} \rightarrow 1\text{-GEN}$ already over RCA₀, thereby resolving these questions. In particular, for any two distinct fixed integers $m, n \geq 2$, the principles $m\text{IP}$ and $n\text{IP}$ are equivalent over RCA₀, and hence also over RCA₀ + $B\Sigma_2^0$.

The proof replaces the priority-based labeling argument by a cumulative system of upward-closed supports. The structural labels, stage-truncated halting tests, and jump-decoding strategy originate in Cholak, Downey, and Igusa [1]. The new ingredient is the cumulative, injury-free

2020 *Mathematics Subject Classification.* Primary 03B30; Secondary 03F35, 03D25, 03D32.

Key words and phrases. Reverse mathematics, finite intersection principle, pairwise intersection principle, Cohen 1-genericity, RCA₀.

organization of the supports. A projection lemma and a finite Helly property make pairwise compatibility sufficient for finite compatibility. When the path decoded from a maximal solution fails to be generic, a least *selected* label allows the jump of the original instance to be recovered. The crucial minimum concerns a Σ_1^0 selection predicate, not a Π_2^0 cofinality predicate; thus the entire proof remains within the base theory.

Theorem 1.1. *Over RCA_0 ,*

$$2\text{IP} \longrightarrow 1\text{-GEN}.$$

Consequently,

$$\text{RCA}_0 \vdash \text{FIP} \longleftrightarrow 2\text{IP} \longleftrightarrow 1\text{-GEN}.$$

1.1. Conventions and representations. We work in RCA_0 with an arbitrary set parameter X . All references to computability and computable enumerability are relative to X , unless otherwise stated.

We write $2^{<\mathbb{N}}$ for the finite binary strings, λ for the empty string, and $\sigma \preceq \tau$ for the (not necessarily proper) prefix relation. For a string ν and a c.e. predicate E on strings, let

$$C_\nu = \{\tau : \nu \preceq \tau\}, \quad \uparrow E = \{\tau : (\exists \rho \preceq \tau) \rho \in E\}.$$

The supports will be upward-closed classes of finite strings. Thus a nonempty intersection has a finite witness.

Essential formal convention. The notation $W_i, K, V_{i,j}$ and the notation for c.e. supports denote predicates equipped with enumerations. We do not assume that these classes exist as second-order sets. Operations on supports abbreviate $\Sigma_1^0(X)$ formulas. The actual sets to which we shall apply 2IP will be uniformly computable in X and will be constructed in Section 4.

Fix a universal listing $(W_i)_{i \in \mathbb{N}}$ of the X -c.e. predicates on strings, with finite, monotone approximations $W_{i,s}$, normalized so that

$$(1) \quad W_{i,s} = \emptyset \quad (i > s), \quad \rho \in W_{i,s} \implies |\rho| \leq s.$$

This normalization is obtained by delaying enumerations and does not change any W_i .

Let $K_s = K_s^X$ be the finite monotone approximation to the jump predicate. We write

$$j \in K \iff (\exists s) j \in K_s.$$

Fix an index j_∞ of a program that never halts, uniformly for every oracle.

For a path $Y \in 2^\mathbb{N}$, we say that Y *meets* W_i if one of its prefixes belongs to W_i . We say that Y *avoids* W_i if

$$(\exists n) C_{Y \upharpoonright n} \cap \uparrow W_i = \emptyset.$$

This meet-or-avoid alternative is equivalent to the usual genericity condition. Indeed, if Y does not meet W_i , avoidance is equivalent to the existence of a prefix of Y with no extension in W_i .

2. CUMULATIVE SUPPORTS

The central construction assigns to each label an upward-closed support of binary strings. A finite snapshot records requirements already witnessed along a scope, while a truncated component retains the information needed for the later jump decoding.

For every string ν , we record the positive hits already visible at stage $|\nu|$:

$$(2) \quad M_\nu = \{i : (\exists \rho \preceq \nu) \rho \in W_{i,|\nu|}\}.$$

By (1), $M_\nu \subseteq \{0, \dots, |\nu|\}$. Thus M_ν is a finite set with a code computable in X . Moreover,

$$(3) \quad \nu \preceq \mu \implies M_\nu \subseteq M_\mu.$$

Define the truncated component

$$(4) \quad V_{i,j} = \{\rho : (\exists s) [\rho \in W_{i,s} \wedge j \notin K_s]\}.$$

We have

$$(5) \quad j \notin K \implies V_{i,j} = W_i;$$

$$(6) \quad j \in K_h \implies (\forall \rho \in V_{i,j}) |\rho| < h.$$

For the second implication, every stage witnessing $\rho \in V_{i,j}$ precedes h , by monotonicity of K_s .

We use structural labels A_σ , for every string σ , and coding labels $B_{e,j,\nu}$, for every e, j, ν . Their supports are

$$\mathcal{U}(A_\sigma) = C_\sigma$$

and

$$(7) \quad \mathcal{U}(B_{e,j,\nu}) = U_{e,j,\nu} = C_\nu \cap \left(\bigcup_{\substack{i < e \\ i \notin M_\nu}} \uparrow W_i \cup T_{e,j,\nu} \right),$$

where

$$T_{e,j,\nu} = \begin{cases} \uparrow V_{e,j}, & e \notin M_\nu, \\ \emptyset, & e \in M_\nu. \end{cases}$$

The *scope*, namely the prefix delimiting the support, is

$$\text{scope}(A_\sigma) = \sigma, \quad \text{scope}(B_{e,j,\nu}) = \nu.$$

Every support is upward closed under extension and contained in the cone of its scope.

All components $i < e$ are untruncated. Only the current component $i = e$ depends on the stage at which j enters K . There are no root-label shifts or injury actions.

3. PROJECTION, NESTING, AND THE HELLY PROPERTY

We now establish the structural properties of the supports. Projection brings them to a common cone, where nesting turns pairwise compatibility into a common intersection. The argument must remain valid for internally finite lists in RCA_0 .

Lemma 3.1 (Projection). *If $\nu \preceq \mu$, then*

$$U_{e,j,\nu} \cap C_\mu = C_\mu \quad \text{or} \quad U_{e,j,\nu} \cap C_\mu = U_{e,j,\mu}.$$

There is a flag $q(e, j, \nu, \mu)$ decidable in X , using data up to stage $|\mu|$, that selects one of the two representations: if q holds, the first identity holds; if q fails, the second identity holds.

The flag need not decide whether $U_{e,j,\mu}$ itself equals all of C_μ .

Proof. Put $u = |\mu|$ and define

$$\begin{aligned} q(e, j, \nu, \mu) \iff & (\exists i < e) [i \in M_\mu \setminus M_\nu] \\ & \vee \left[e \in M_\mu \setminus M_\nu \wedge \right. \\ & \left. (\exists s \leq u)(\exists \rho \preceq \mu) [\rho \in W_{e,s} \wedge j \notin K_s] \right]. \end{aligned}$$

If the first disjunct holds, an untruncated component W_i present in the old support already contains a prefix of μ . The old support therefore contains all of C_μ .

If the second disjunct holds, the same conclusion follows from the presence of a prefix of μ in $V_{e,j}$.

Now suppose that q is false. For every $i < e$, the exclusions in the two snapshots coincide, by (3). Hence the lower components of the two supports, restricted to C_μ , are identical.

If $e \in M_\nu$, the current component is omitted from both supports. If $e \notin M_\mu$, the same $V_{e,j}$ appears in both.

It remains to consider the case

$$e \in M_\mu \setminus M_\nu.$$

If $j \notin K_u$, the prefix of μ witnessing $e \in M_\mu$ belongs to $V_{e,j}$ at stage u , contradicting $\neg q$. Thus $j \in K_u$.

Every generator ρ of $V_{e,j}$ now has length less than u , by (6). If such a generator were compatible with μ , it would be a prefix of μ ; its enumeration stage, which is less than u , would make q true. Consequently,

$$C_\mu \cap \uparrow V_{e,j} = \emptyset.$$

The current component of the old support is empty on this cone, while that of the new support is omitted. This yields the second identity. $\square$

Lemma 3.2 (Nesting at a fixed scope). *For every μ :*

(1) *if $e < d$, then*

$$U_{e,j,\mu} \subseteq U_{d,k,\mu} \quad \text{for all } j, k;$$

(2) *for fixed e , the supports $U_{e,j,\mu}$ are linearly ordered by inclusion.*

Proof. The first inclusion follows from $V_{e,j} \subseteq W_e$: every component of the support with index e appears in full in the support with index d , with the same exclusions imposed by M_μ .

For the second assertion, if $e \in M_\mu$, the supports coincide for all j , since the current component is omitted.

Suppose therefore that $e \notin M_\mu$. If neither j nor k ever enters K , then

$$V_{e,j} = V_{e,k} = W_e.$$

Otherwise consider the first entry of either of them. If j enters first, every stage preceding its entry also precedes the entry of k , or k never enters. Therefore $V_{e,j} \subseteq V_{e,k}$. The opposite case is symmetric.

The first entry stage, when it exists, is a Σ_1^0 least number. $\square$

Lemma 3.3 (Finite Helly property in RCA_0). *Every internally finite family of labels with nonempty supports that intersect pairwise has nonempty total intersection.*

Proof. We may remove repetitions. The empty case, with the usual convention for an empty intersection, and the case of a single label are immediate.

For a family of at least two labels, their scopes are pairwise comparable, because the supports are contained in the cones of their scopes. Choose a scope μ of maximum length.

Every support has nonempty intersection with C_μ : this is witnessed by its intersection with a label whose scope is μ ; for that label itself, nonemptiness suffices.

Restrict all supports to C_μ . The structural supports become C_μ . For the others, compute the flag from Lemma 3.1. True flags again yield C_μ ; false flags yield supports $U_{e,j,\mu}$.

If there are no false flags, we are done. Otherwise choose the least e among the finitely many labels with a false flag, by bounded search. Among those with this e , consider the finite list of their indices j .

If one of these indices enters K , take the first stage at which at least one of them enters, and choose an index that enters then. This is just one Σ_1^0 least entry stage, not a choice of a least element in an inclusion ordering.

If none enters, choose any label from the list: all the corresponding $V_{e,j}$ coincide with W_e . The possible exclusion $e \in M_\mu$ is common to all supports in the list.

In either case, the chosen restricted support is included in those with the same e , and, by Lemma 3.2, in those with a larger index. It is also included in the full cones. It is nonempty by the second paragraph; a witness in it therefore belongs to every support.

The bounded choices concern decidable finite data. The only possible unbounded choice is the first entry of an element of the list into K , which is permitted by $I\Sigma_1^0$. Thus the argument also works for families of internally nonstandard finite cardinality. $\square$

4. REALIZATION AS A COMPUTABLE FAMILY

The preceding combinatorial assertions concern enumerable classes of strings. We now code them as an actual family of sets: common odd tokens record intersection witnesses, and private even markers identify the family members. The coding uses no arithmetical comprehension and excludes empty sets.

Fix codes for all potential labels and a uniform enumeration of positive assertions $\tau \in \mathcal{U}(L)$. We use finite monotone approximations $\mathcal{U}_s(L)$, decidable in X , with only finitely many labels and strings examined at each stage. Every true membership assertion appears at some stage.

This enumeration is obtained by dovetailing formulas (4) and (7), with restrictions on the codes at the current stage. The truncation in $V_{i,j}$ concerns the stage witnessing $\rho \in W_{i,s}$, as in the definition, and not a later stage at which the witness is processed.

Enumerate without repetitions only those labels for which a witness to nonemptiness has appeared. Let

$$L_0, L_1, L_2, \dots$$

be the resulting enumeration.

All structural labels A_σ have nonempty support and are inserted according to a fixed computable scan. Consequently, RCA_0 proves that each L_n exists: the first $n + 1$ distinct structural witnesses give a finite bound for the appearance of at least $n + 1$ labels. The function $n \mapsto L_n$ is total and computable in X .

Define a family

$$\mathcal{A} = (D_n)_{n \in \mathbb{N}}$$

of subsets of $\mathbb{N}$. The set D_n contains the single even number $2n$. For odd numbers, let

$$(8) \quad 2\langle \tau, s \rangle + 1 \in D_n \iff \begin{cases} L_n \text{ has already been indexed by stage } s, \\ \tau \in \mathcal{U}_s(L_n). \end{cases}$$

The two conditions on the right are required jointly.

This test is decidable in X : one need only simulate the finite stage s . If fewer than $n + 1$ labels have appeared by that stage, the answer is negative. Thus the relation $z \in D_n$ defines a set by $\Delta_1^0(X)$ comprehension.

All the D_n are nonempty and pairwise distinct, thanks to the private markers $2n$. For two distinct indices, a common element must be odd; hence

$$(9) \quad D_n \cap D_m \neq \emptyset \iff \mathcal{U}(L_n) \cap \mathcal{U}(L_m) \neq \emptyset.$$

The same equivalence holds for every finite family of at least two distinct indices. In one direction, a common odd element gives a witness τ . Conversely, given a common witness τ for the supports, $B\Sigma_1^0$, available in RCA_0 , collects the finitely many required enumeration and indexing stages. A later common stage provides the shared odd number in (8).

For a single label, both sides of the equivalence are true by construction. Thus the Helly property transfers to the actual family $\mathcal{A}$.

Applying 2IP to $\mathcal{A}$, we obtain a maximal subfamily. We may represent it by a total function f such that

$$(D_{f(r)})_{r \in \mathbb{N}}$$

is maximal with the pairwise intersection property.

If the formulation of the principle instead returns a sequence of sets (E_r) , we recover $f(r)$ by searching for the unique n with $2n \in E_r$. The search terminates because each E_r is a member of $\mathcal{A}$. We do not decide equality between arbitrary sets.

For a potential label L , define

$$(10) \quad \text{Sel}(L) \iff (\exists r) L_{f(r)} = L.$$

This is a $\Sigma_1^0(X \oplus f)$ property. We do not assume that the range of f exists as a set. A label with empty support is not indexed and is not selected.

By (9), two selected labels have intersecting supports. Conversely, if a label has nonempty support and is compatible with every selected label, its set can be added to the subfamily while preserving 2IP. Maximality therefore forces it to be selected.

Formally, to add D_n one merely prepends it to the sequence $(D_{f(r)})_r$. This operation is available in RCA_0 and uses the extensional maximality of the subfamily.

5. THE PATH AND THE CHARACTERIZATION OF SELECTION

The structural labels selected by a maximal pairwise-intersecting subfamily determine an infinite path through the binary tree. We then characterize the selection of any label by the cofinality of its support along that path.

Lemma 5.1 (Structural path). *From $X \oplus f$ one can compute a unique path $Y \in 2^{\mathbb{N}}$ such that*

$$\text{Sel}(A_\sigma) \iff \sigma \preceq Y.$$

The argument can be formalized in RCA_0 .

Proof. The support of A_λ intersects every nonempty support. Therefore A_λ is selected by maximality. The scopes of selected structural labels are pairwise comparable.

Suppose that A_σ is selected and neither of its two children $A_{\sigma \smallfrown 0}$, $A_{\sigma \smallfrown 1}$ is selected. By maximality, there are selected labels H_0, H_1 with

$$\mathcal{U}(H_b) \cap C_{\sigma \smallfrown b} = \emptyset \quad (b = 0, 1).$$

Indeed, if a child had no obstruction among the selected labels, it could be added to the solution.

The three labels A_σ, H_0, H_1 have nonempty total intersection, by Lemma 3.3. Let τ be a common witness.

Since the supports are upward closed, we can extend τ to a string of length greater than $|\sigma|$. That extension belongs to the cone of one of the two children, contradicting the corresponding disjointness condition.

Thus at least one child is selected. They cannot both be selected, since their cones are disjoint.

The formula

$$(\exists \sigma) [|\sigma| = n \wedge \text{Sel}(A_\sigma)]$$

is $\Sigma_1^0(X \oplus f)$. Σ_1^0 induction gives a selected node at every level. It is unique, and the nodes are compatible.

By searching the enumeration for the node at the required level, we compute the bits of Y . The totality just proved permits $\Delta_1^0(X \oplus f)$ comprehension. Uniqueness and comparability give both directions of the equivalence in the statement. $\square$

Definition 5.2. A support $\mathcal{U}(L)$ is *cofinal along Y* if

$$(11) \quad \text{Cof}_Y(L) \iff (\forall n)(\exists \tau \succeq Y \upharpoonright n) \tau \in \mathcal{U}(L).$$

This is a $\Pi_2^0(X \oplus Y)$ property.

Lemma 5.3 (Cofinality and maximality). *For every potential label L ,*

$$(12) \quad \boxed{\text{Sel}(L) \iff \text{Cof}_Y(L)}.$$

Proof. If L is selected, its support intersects every $C_{Y \upharpoonright n}$, since all the $A_{Y \upharpoonright n}$ are selected.

Conversely, suppose $\text{Cof}_Y(L)$. Its scope is a prefix of Y : if it were incompatible with a prefix of Y , it could not be cofinal.

The same observation applies to every selected label H , by the direction already proved. Let μ be the longer of the two scopes. The restrictions of $\mathcal{U}(L)$ and $\mathcal{U}(H)$ to C_μ are both nonempty.

By projection and nesting, these restrictions are comparable by inclusion, including the case of the full cone. Thus they intersect.

The label L has nonempty support and is therefore indexed. It is compatible with every selected label, and maximality forces it into the solution.

This also deals with potential labels that are not indexed: for them both properties are false. $\square$

Remark. Two arbitrary open sets may both be cofinal along the same path without intersecting. Here the implication is justified by projection to a common scope and nesting. This is a specific property of the family (7).

6. STABILIZATION OF POSITIVE HITS ONLY

Lemma 6.1 (Exact finite snapshot). *For every a and every path Y , there exists $t > a$ such that, putting $\nu = Y \upharpoonright t$, we have, for each $i \leq a$,*

$$(13) \quad i \in M_\nu \iff \text{Hit}(i, Y), \quad \text{Hit}(i, Y) \equiv (\exists \rho \preceq Y) \rho \in W_i.$$

This is provable in RCA_0 .

Proof. Let

$$c(s) = \#\{i \leq a : (\exists \rho \preceq Y \upharpoonright s) \rho \in W_{i,s}\}.$$

The function c is computable in $X \oplus Y$, nondecreasing and bounded by $a + 1$.

The predicate

$$P(r) \equiv (\exists s) [a + 1 - c(s) = r]$$

is $\Sigma_1^0(X \oplus Y)$ and nonempty. The Σ_1^0 least number principle, available from $I\Sigma_1^0$, gives its least element r_0 .

A stage s_0 witnessing $P(r_0)$ achieves the maximum of c . By monotonicity,

$$c(s) = c(s_0) \quad (s \geq s_0).$$

No new index $i \leq a$ can therefore be recorded subsequently.

Choose $t > \max\{a, s_0\}$. If W_i contains a prefix ρ of Y , then at a sufficiently late stage that prefix appears in $W_{i,s}$ and is a prefix of $Y \upharpoonright s$. Thus index i must already have been present in the snapshot at t .

The reverse implication in (13) is immediate. $\square$

Remark. A bounded sequence of positive events is stabilized: each index $i \leq a$ can contribute just one new event. We do not collect a family of bounds for components that are individually noncofinal. The latter operation will be replaced by a single bound for a cumulative support.

7. THE LEAST SELECTED INDEX AND DECODING THE JUMP

Suppose that the path Y from Lemma 5.1 is not 1-generic relative to X . Then there is an index a such that Y neither meets nor avoids W_a . In particular,

$$(14) \quad \neg \text{Hit}(a, Y), \quad (\forall n) C_{Y \upharpoonright n} \cap \uparrow W_a \neq \emptyset.$$

Apply Lemma 6.1 through index a and fix $\nu = Y \upharpoonright t$ with an exact snapshot.

Every residual component W_i with $i \leq a$ and $i \notin M_\nu$ therefore misses every prefix of Y .

For $r \leq a$, define the potential labels

$$(15) \quad N_r = B_{r, j_\infty, \nu}.$$

Since $j_\infty \notin K$, their supports are

$$(16) \quad \mathcal{U}(N_r) = C_\nu \cap \bigcup_{\substack{i \leq r \\ i \notin M_\nu}} \uparrow W_i.$$

By (14), $a \notin M_\nu$ and $\mathcal{U}(N_a)$ is cofinal along Y . By Lemma 5.3, N_a is selected.

Therefore choose

$$(17) \quad \boxed{e = \min\{r \leq a : \text{Sel}(N_r)\}}.$$

This choice uses the least number principle for a $\Sigma_1^0(X \oplus f)$ predicate, by (10). We are not taking the least index satisfying the Π_2^0 cofinality property of the individual W_r .

Now let

$$(18) \quad \mathcal{R} = C_\nu \cap \bigcup_{\substack{i < e \\ i \notin M_\nu}} \uparrow W_i.$$

If $e = 0$, then $\mathcal{R} = \emptyset$. If $e > 0$, we have the exact identity

$$(19) \quad \mathcal{R} = \mathcal{U}(N_{e-1}).$$

In the latter case, where $e > 0$, minimality gives $\neg \text{Sel}(N_{e-1})$. Lemma 5.3 then implies that $\mathcal{R}$ is not cofinal.

For clarity, note also that

$$(20) \quad e \notin M_\nu.$$

Indeed, if $e = 0$ and $e \in M_\nu$, then $\mathcal{U}(N_e) = \emptyset$, contradicting its selection. If $e > 0$ and $e \in M_\nu$, the current component of N_e is omitted, and hence $\mathcal{U}(N_e) = \mathcal{U}(N_{e-1})$. The equivalence between cofinality and selection would then contradict the minimality of e .

In either case, there exists a *single* level b such that

$$(21) \quad C_{Y \upharpoonright b} \cap \mathcal{R} = \emptyset.$$

This is an existence assertion, not a claim of uniqueness of b , and no bound is computed from the solution. In particular, no separate bound is chosen for each $i < e$.

Lemma 7.1 (A column decoding K). *For the e, ν just chosen and every j ,*

$$(22) \quad \boxed{\text{Sel}(B_{e,j,\nu}) \iff j \notin K.}$$

Proof. If $j \notin K$, then by (5) and (16),

$$U_{e,j,\nu} = \mathcal{U}(N_e).$$

This support is cofinal because N_e is selected. Therefore $B_{e,j,\nu}$ is selected by Lemma 5.3.

Now suppose $j \in K_h$. The support $U_{e,j,\nu}$ consists of the part $\mathcal{R}$ and the current component, when present.

The latter is generated by strings $\rho \in V_{e,j} \subseteq W_e$ of length less than h . If this component is present, then $e \notin M_\nu$, and the exact snapshot guarantees that none of these strings is a prefix of Y .

Each ρ is therefore incompatible with $Y \upharpoonright (h+1)$: since its length is less than h , if it were compatible it would be a prefix. Consequently,

$$C_{Y \upharpoonright (h+1)} \cap U_{e,j,\nu} = \emptyset.$$

If the component is omitted, the same equality is obvious.

Together with (21), this gives

$$C_{Y \upharpoonright \max\{b, h+1, t\}} \cap U_{e,j,\nu} = \emptyset.$$

The label is not cofinal and therefore is not selected. □

By (22), the jump predicate has two complementary definitions:

$$j \in K \iff (\exists s) j \in K_s, \quad j \notin K \iff \text{Sel}(B_{e,j,\nu}).$$

Both right-hand sides are $\Sigma_1^0(X \oplus f)$, with the numerical parameters e, ν fixed.

Thus Δ_1^0 comprehension produces a set J with

$$(23) \quad j \in J \iff (\exists s) j \in K_s.$$

This is exactly X' .

Computationally, one alternates between searching for the entry of j into K and searching for label $B_{e,j,\nu}$ in the enumeration of the solution. Exactly one of these two searches terminates.

The decoder need not know or compute the bound b : that bound is used only in the correctness proof.

8. PROOF OF THE MAIN THEOREM AND CONSEQUENCES

Lemma 8.1. *In RCA_0 , if the jump X' exists as a set, then there exists a 1-generic relative to X .*

Proof. Construct an increasing sequence of strings $(\sigma_s)_s$, with $\sigma_0 = \lambda$. At stage s , use X' to decide the $\Sigma_1^0(X)$ question

$$(\exists \tau \succeq \sigma_s)(\exists \rho \preceq \tau) \rho \in W_s.$$

If the answer is positive, search for witnesses τ, ρ and put

$$\sigma_{s+1} = \tau \frown 0.$$

If the answer is negative, put

$$\sigma_{s+1} = \sigma_s \frown 0.$$

The jump decides the question uniformly, using the index of an X -oracle program that searches for witnesses. The search on the positive branch terminates by correctness of the decision.

Totality of the construction at all stages is proved by $\Sigma_1^0(X \oplus X')$ induction on the existence of finite computation traces. There is no need to include in the inductive formula the assertion that all preceding requirements have been satisfied.

Since $|\sigma_s| \geq s$, the union of the strings defines a set G by $\Delta_1^0(X \oplus X')$ comprehension.

At stage s , the positive branch meets W_s , since $\rho \preceq G$. The negative branch certifies

$$C_{\sigma_s} \cap \uparrow W_s = \emptyset.$$

Therefore G meets or avoids every W_s and is thus 1-generic relative to X . □

Proof of Theorem 1.1. Let X be arbitrary. Construct the family computable in X from Section 4 and apply 2IP.

From the maximal solution we obtain f , and from Lemma 5.1 we obtain the path Y .

If Y is 1-generic relative to X , we already have the desired witness.

If it is not, choose a requirement W_a that Y neither meets nor avoids. Lemma 6.1 and the construction in Section 7 produce the set X' in RCA_0 . Lemma 8.1 then produces a 1-generic relative to X .

In either case, a witness to 1-GEN exists. Since X was arbitrary,

$$\text{RCA}_0 \vdash 2\text{IP} \rightarrow 1\text{-GEN}.$$

Proposition 5.4 of [1] gives $\text{RCA}_0 \vdash 1\text{-GEN} \rightarrow \text{FIP}$, while their Proposition 5.6 gives $\text{RCA}_0 \vdash \text{FIP} \rightarrow 2\text{IP}$. Together with the implication just proved, these form a cycle, and therefore

$$\text{RCA}_0 \vdash \text{FIP} \longleftrightarrow 2\text{IP} \longleftrightarrow 1\text{-GEN}.$$

□

Corollary 8.2. *For each fixed integer $n \geq 2$, RCA_0 proves that $n\text{IP}$, 2IP , FIP , and 1-GEN are equivalent. In particular, for every fixed $m, n \geq 2$,*

$$\text{RCA}_0 \vdash m\text{IP} \longleftrightarrow n\text{IP}.$$

Proof. Combine Theorem 1.1 with the known implications

$$\text{FIP} \rightarrow n\text{IP} \rightarrow 2\text{IP}, \quad \text{FIP} \leftrightarrow 1\text{-GEN},$$

recalled in [1, Propositions 5.4–5.6]. □

9. FORMALIZATION IN THE BASE THEORY

The proof is intended literally as a derivation in RCA_0 , not merely as an argument valid in its ω -models. For reference, the following table identifies the resources involved at the potentially sensitive steps.

Step	Resource in RCA_0
Snapshot M_ν and projection flag	Finite data decidable in X .
Actual family (D_n)	$\Delta_1^0(X)$ comprehension and stage coding.
Helly property for an internally finite list	Bounded search and, if needed, a least Σ_1^0 entry stage.
Witnesses for shared support elements	$B\Sigma_1^0$ bounding.
Path from f	$\Sigma_1^0(X \oplus f)$ induction and Δ_1^0 comprehension.
Stabilization up to a	Σ_1^0 minimization of the reversed values of a bounded counting function.
Choice of e	$\Sigma_1^0(X \oplus f)$ minimization for selection.
Lower bound b	Noncofinality of one support, $\mathcal{U}(N_{e-1})$, when $e > 0$.
Existence of X'	Two complementary enumerations and Δ_1^0 comprehension.
Generic from X'	Computable recursion and $\Sigma_1^0(X \oplus X')$ induction.

The minimum in (17) uses information provided by the solution: the positive appearance of a label in its enumeration.

The preceding support N_{e-1} collects all lower components into one class. Its nonselection gives a single avoidance level. This is the mechanism that replaces the choice of a Π_2^0 minimum and the collection of bounds for lower components.

The choices of a, t, e, ν are existential and nonuniform. We do not assert the existence of a single algorithm that determines, from a solution, which of the two cases Y is generic or X' is decodable holds, or that uniformly finds the parameters of the decoder.

Thus the conclusion concerns the implication over RCA_0 . It does not automatically provide a Weihrauch reduction from 1-GEN to FIP.

10. CONCLUDING REMARKS

Theorem 1.1 answers Question 1 of Cholak–Downey–Igusa [1] affirmatively: the pairwise and finite intersection principles are equivalent already over RCA_0 . Corollary 8.2 also answers their Question 2, in its RCA_0 formulation, for every pair of fixed integers $m, n \geq 2$. Since these equivalences are provable already in RCA_0 , they also hold over $\text{RCA}_0 + B\Sigma_2^0$, answering the second part of Question 2 as well. Moreover, the Σ_2^0 -induction hypothesis in the known derivation of $2\text{IP} \rightarrow 1\text{-GEN}$ is not needed for the implication.

The argument relies on two features of the new supports: their finite Helly property, which is available for internally finite lists, and the cumulative lower part of a least selected label, which admits a single avoidance level. The actual family is $\Delta_1^0(X)$, and its private markers allow solutions to be decoded without requiring the range of an enumerating function as a second-order set.

The proof is nonuniform: the choice between an already generic path and a decodable jump, as well as the parameters of the jump decoder, is existential. The equivalence over RCA_0 does not by itself establish a Weihrauch reduction between the corresponding problems.

In fact, a companion manuscript [2] establishes

$$1\text{-GEN} \not\leq_W \text{FIP}.$$

Thus, although RCA_0 proves that FIP implies 1-GEN, this implication cannot be realized by a Weihrauch reduction in the representations considered there.

ACKNOWLEDGMENTS

The main result was discovered by ChatGPT 6 Pro (OpenAI), which was also used in developing the proof and preparing the manuscript. A preliminary version of the proof was reviewed using Claude Opus 5.5 (Anthropic), and a later version underwent a separate mathematical audit using Astra 6.1 Ultra (OpenAI). The author takes full responsibility for the correctness of the mathematical statements and proofs, for the accuracy of the citations and attributions, and for the final content of the manuscript.

REFERENCES

- [1] P. Cholak, R. G. Downey, and G. Igusa, *Any FIP real computes a 1-generic*, Trans. Amer. Math. Soc. **369** (2017), no. 8, 5855–5869. <https://arxiv.org/abs/1502.03785>.
- [2] P. Cintioli, *1-Genericity is not Weihrauch Reducible to the Finite Intersection Principle*, manuscript, 2026.
- [3] D. Diamondstone, R. Downey, N. Greenberg, and D. Turetsky, *The finite intersection principle and genericity*, Math. Proc. Cambridge Philos. Soc. **160** (2016), no. 2, 279–297.
- [4] D. D. Dzhafarov and C. Mummert, *On the strength of the finite intersection principle*, Israel J. Math. **196** (2013), no. 1, 345–361. <https://arxiv.org/abs/1109.3374>.
- [5] D. D. Dzhafarov and C. Mummert, *Reverse Mathematics: Problems, Reductions, and Proofs*, Theory and Applications of Computability, Springer, Cham, 2022.
- [6] S. Sanders, *Refining the taming of the reverse mathematics zoo*, Notre Dame J. Formal Logic **59** (2018), no. 4, 579–597. <https://arxiv.org/abs/1602.02270>.

MATHEMATICS DIVISION, SCHOOL OF SCIENCE AND TECHNOLOGY, UNIVERSITY OF CAMERINO, 62032 CAMERINO, ITALY

Email address: patrizio.cintioli@unicam.it