

CONE AVOIDANCE AND THE NO-LEAST-JOIN THEOREM

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ABSTRACT. We give a column-avoidance modification of the no-least-join construction of Downey, Greenberg, Lewis, and Montalbán. We show that if C is computably enumerable, $A, B <_T C$, and $A \not\leq_T B$, then for every noncomputable set E there is a set $X \leq_T C$ such that $B \leq_T A \oplus X$ and $E \not\leq_T X$. Consequently, for the corresponding Turing degrees a, b, c , the degrees $x \leq c$ satisfying $b \leq a \vee x$ have no nonzero common lower bound; equivalently, no nonzero Turing cone contains all of them. This gives a negative answer to their Question 1.19.

1. INTRODUCTION

Downey, Greenberg, Lewis, and Montalbán [DGLM13] study extensions of embeddings of finite upper semilattices below computably enumerable Turing degrees, as part of their analysis of the two-quantifier theory of the Δ_2^0 degrees with join. One of the objects isolated in their work is the following upward-closed family. For degrees $a, b \leq c$, let

$$(1) \quad \mathcal{F}_{a,b}^c = \{x \leq c : b \leq a \vee x\}.$$

Following their terminology, we call this the *restricted difference filter*. The word “filter” refers only to upward closure inside the initial segment below c .

The behavior of these filters already exhibits a sharp contrast. DGLM’s Theorem 1.11 gives c.e. degrees

$$\mathbf{0} < a < b < c$$

for which $\mathcal{F}_{a,b}^c$ is exactly the cone above b , restricted below c . On the other hand, their no-least-join theorem, Theorems 1.23 and 6.1 of [DGLM13], shows that if c is c.e., $a, b < c$, $a \not\leq b$, and $b > \mathbf{0}$, then

$$\mathcal{F}_{a,b}^c \not\subseteq \{x : b \leq x\}.$$

Thus, under these hypotheses, the degree b itself is not information that every witness to the cupping requirement must compute.

A natural strengthening is to ask whether *any* fixed nonzero degree can be unavoidable. Equivalently, can a restricted difference filter $\mathcal{F}_{a,b}^c$ with $a \not\leq b$ be contained in some nonzero Turing cone? This question also arises from a second phenomenon discovered in [DGLM13]. Their Theorem 1.18 constructs pairwise incomparable c.e. degrees a, b, d, e below a c.e. degree c such that

$$\mathcal{F}_{a,b}^c \subseteq \mathcal{F}_{d,e}^c.$$

Hence one cupping requirement can force another. DGLM ask whether the information shared by all witnesses to the first requirement might itself compute a fixed nonzero degree. They formulate the question as follows.

Question 1.19 [DGLM13]. Is there a c.e. set C , incomparable sets $A, B <_T C$ and a noncomputable set E such that for any Δ_2^0 set X which is computable in C , if X joins A above B , then X computes E ?

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In Turing-reducibility notation, “ X joins A above B ” means $B \leq_T A \oplus X$, so Question 1.19 asks whether one can find a c.e. set C , incomparable sets $A, B <_T C$, and a noncomputable set E such that

$$(2) \quad (\forall X \leq_T C) [B \leq_T A \oplus X \implies E \leq_T X].$$

The additional requirement that X be Δ_2^0 is automatic from $X \leq_T C$, since C is c.e. At the level of degrees, Question 1.19 therefore asks exactly whether, in the incomparable case, a restricted difference filter can have a nonzero common lower bound.

We show that this cannot happen. In fact, no nonzero Turing cone contains the whole restricted difference filter: for every prescribed nonzero degree there is a witness to the cupping requirement which does not compute it.

Theorem 1.1 (Cone avoidance). *Let C be c.e., let $A, B <_T C$, and suppose that $A \not\leq_T B$. For every noncomputable set E there is a set $X \leq_T C$ such that*

$$B \leq_T A \oplus X \quad \text{and} \quad E \not\leq_T X.$$

Corollary 1.2. *Let c be a c.e. degree, and let $a, b < c$ with $a \not\leq b$. Then $\mathbf{0}$ is the only common lower bound of $\mathcal{F}_{a,b}^c$. In particular, Question 1.19 of [DGLM13] has a negative answer.*

Proof. For every $e > \mathbf{0}$, Theorem 1.1, applied to representatives, gives an $x \in \mathcal{F}_{a,b}^c$ with $e \not\leq x$. Thus every common lower bound is $\mathbf{0}$, and $\mathbf{0}$ itself is a lower bound. The incomparability in Question 1.19 implies the hypothesis $a \not\leq b$. $\square$

Equivalently, whenever $a \not\leq b$, the restricted difference filter $\mathcal{F}_{a,b}^c$ is contained in no nonzero Turing cone. Thus the no-least-join theorem, which excludes b itself as unavoidable information, extends to cone avoidance for every prescribed nonzero degree.

This also clarifies the dependence phenomenon of Theorem 1.18 of [DGLM13]. Although membership in one restricted difference filter may force membership in another, this implication cannot, under the hypotheses of Question 1.19, be explained by a fixed nonzero degree lying below every witness to the first cupping requirement. In this precise sense, the dependence exhibited by Theorem 1.18 is genuinely relational.

The order of quantifiers is important. The witness x can depend on e . We do not construct a fixed pair of members of $\mathcal{F}_{a,b}^c$ having no nonzero common lower bound, and we do not assert a uniform choice of a successful witness.

The proof is an adaptation of the two-output construction in [DGLM13, Section 6]. If $E \not\leq_T B$, then $X = B$ already works. In the remaining case, put $D = B \oplus E \equiv_T B$. We continue to code the *whole* of D into each $A \oplus X_j$, but use only the odd column of D for diagonalization. Merely replacing B by E in the original no-least-join theorem would not suffice: it would yield a reduction of E , rather than the required reduction of B , from $A \oplus X$.

We therefore isolate the part of the construction that does not depend on its diagonalization target. The structural transfer theorem below retains the scheduling, coding, and safety mechanism of [DGLM13, Section 6], while allowing any finite effective rule for certificates that obeys the stated structural conditions. Its proof is included, rather than treating the transfer as a consequence of the statement of the original theorem.

Question 1.19 arises in DGLM’s broader study of the extension-of-embeddings problem. The present result does not settle the sufficiency of the anti-cupping condition, nor does it decide the two-quantifier theory of the Δ_2^0 degrees with join. It does, however, rule out one natural form that an additional obstruction might take: a fixed nonzero computational core shared by all witnesses to a single cupping requirement.

2. APPROXIMATIONS, STRINGS, AND HISTORICAL RECORDS

This section fixes the approximation and bookkeeping conventions used throughout the construction. In particular, we isolate a monotone approximation to a common modulus, and we introduce the freshness and frozen-budget mechanisms that will control historical axioms. Conventions. All sets are subsets of ω and are identified with their characteristic functions. We write $\leq_T$ and $\equiv_T$ for Turing reducibility and Turing equivalence, $\oplus$ for the usual join, and $\mathbf{0}$ for the computable degree. When degree notation is in use, lower-case letters such as a, b, c, e denote Turing degrees. Standard background and terminology in computability theory may be found in [Soa16].

Throughout the construction, A and D have computable approximations (A_s) and (D_s) . When $A, D \leq_T C$ for a c.e. set C , these approximations may be chosen to have a common C -computable convergence bound. Indeed, fix reductions of A and D to C , and approximate them using a computable enumeration (C_s) . With oracle C , the finitely many true computations on arguments at most n and their uses can be found. Waiting until C_s agrees with C below these uses, and until these computations have run, gives a bound after which both approximations are correct through n . Use a fixed default bit for any computation not yet observed to converge.

For $s, n \in \omega$, define

$$(3) \quad m_s(n) = \min\{t \leq s : (\forall u \in [t, s]) (A_u, D_u) \upharpoonright (n+1) = (A_s, D_s) \upharpoonright (n+1)\}.$$

Thus $m_s(n)$ records the beginning of the current uninterrupted interval of stability, not the most recent stage belonging to that interval.

Lemma 2.1. *The approximation (m_s) is computable and converges pointwise to a function m . Each coordinate $m_s(n)$ is nondecreasing in s . Moreover,*

$$(4) \quad m_{s+1} \upharpoonright n \neq m_s \upharpoonright n \iff (A_{s+1}, D_{s+1}) \upharpoonright n \neq (A_s, D_s) \upharpoonright n,$$

and, for every finite string μ of natural numbers,

$$(5) \quad \mu \preceq m_t, \quad \mu \not\preceq m_s, \quad t < s \implies (\forall v \geq s) \mu \not\preceq m_v.$$

If the two set approximations have a common C -computable convergence bound, then $m \leq_T C$.

Proof. A change in $(A_s, D_s) \upharpoonright (n+1)$ resets $m_s(n)$ to the current stage; in the absence of a change, its value remains fixed. This proves monotonicity and convergence. A change below n changes at least the coordinate $n-1$ of m_s , whereas no change below n leaves every coordinate below n fixed. This proves (4), with the case $n = 0$ immediate. Monotonicity proves (5). Finally, given a convergence bound for the first $n+1$ bits of both sets, compute the finite history up to that bound. The start of its final stable interval is $m(n)$. $\square$

String notation. For binary strings, $\sigma \preceq \tau$ means that σ is an initial segment of τ , and $\sigma \prec \tau$ means proper initial segment. The same notation applies to strings of natural numbers and infinite sequences. We write $\sigma \perp \tau$ for incompatibility. For a nonempty binary string θ , $\hat{\theta}$ is obtained by changing its last bit. All string coordinates start at 0.

We enumerate axiom systems Ξ_0, Ξ_1 and Γ_0, Γ_1 . An axiom $\mu \mapsto \eta$ in Ξ_j says that an oracle extending μ produces an output extending η . An axiom $(\sigma, \eta) \mapsto \rho$ in Γ_j has the analogous meaning for two binary oracles. The systems Γ_j will be consistent on all oracles. The systems Ξ_j need only be consistent on the approximating oracles m_s at the relevant stages and on their limit m . Accordingly, before this consistency has been proved, an evaluation of an axiom system means the collection of its applicable output strings. Once these strings have been shown to be comparable, their union is the evaluation. This convention does not presuppose off-path consistency of Ξ_j .

Subscripts s mean that axioms enumerated at stage s are included. For $j < 2$ and $n < \omega$, write

$$\xi_{j,s,n} = \Xi_{j,s}(m_s \upharpoonright n), \quad \xi_{j,s,n}^* = \Xi_{j,s-1}(m_s \upharpoonright n).$$

We also write $\xi_{j,s} = \Xi_{j,s}(m_s)$ and $\xi_{j,s}^* = \Xi_{j,s-1}(m_s)$. Whenever the previous stage evaluations are strings, nonreturn gives

$$(6) \quad \xi_{j,s}^* \preceq \xi_{j,s-1}, \quad \xi_{j,s,n}^* \preceq \xi_{j,s-1,n}.$$

Indeed, an old modulus string that applies at s must still have applied at $s-1$; once such a string ceases to apply, it cannot return.

2.1. Fresh lengths and frozen budgets. We keep a computable increasing historical bound H . It exceeds the stage number and all integers and string lengths recorded so far, and is increased after each finite action. A newly requested *large* integer exceeds the current bound. Every nonempty candidate or certificate string ends in 1, and has length greater than the bound preceding the action in which it is chosen. The bound is then increased to cover all the data of that action.

A *free extension* of a string η is a string $\eta \frown 0^L 1$, where L exceeds the historical bound at the start of its selection. In particular its length exceeds that of every earlier candidate or certificate string.

Lemma 2.2 (Free extensions). *If η' is a free extension of η and ζ is a candidate or certificate string recorded at an earlier stage, then $\zeta \preceq \eta'$ implies $\zeta \preceq \eta$.*

Proof. Otherwise $\eta \prec \zeta \preceq \eta'$. The last bit of ζ is 1, but, by the length bound, its last position lies inside the block of zeros immediately following η in η' . This is impossible. The empty string causes no exception. $\square$

One length choice must *not* be made fresh on every repeated call. For each pair declared correct below, record a *period budget* at the beginning of each uninterrupted period during which it is correct. The budget is greater than its node length and all data recorded before that moment, and is kept fixed throughout that period. At the birth of a new correct node, its budget is chosen after its defining data have been recorded. A break in correctness starts a new budget when correctness resumes. There are only finitely many relevant pairs at each stage, so these records are effective and finite.

This is an explicit implementation of the frozen choice based on the beginning of a correct period in [DGLM13, Section 6.3, Step 3]. Only two properties of these budgets are used: a budget chosen after some historical record exceeds the size of that record; and, during an uninterrupted correct period, repeated ordinary calls use the same budget.

3. THE STRUCTURAL PROTOCOL

We now describe the abstract two-output protocol. The scheduling, coding, and safety mechanism is fixed here independently of the eventual diagonalization target; the only part left open is the certificate rule introduced below.

A procedure p has a requirement label r_p , a modulus string μ_p , a root ρ_p with $|\rho_p| = |\mu_p|$, two candidate strings $\eta_{p,0}, \eta_{p,1}$, and a partial computable map α_p . Its domain is a binary tree above ρ_p : every proper extension of ρ_p in the domain has its immediate predecessor in the domain. We will maintain

$$|\alpha_p(\beta)| = |\beta| \quad (\beta \in \text{dom } \alpha_p)$$

and preservation of the initial-segment relation. Put $a_p = \alpha_p(\rho_p)$.

The active procedures form a finite chain, starting with p_0 . Only the weakest active procedure may call a new procedure. If q is called by p , a node $\beta \in \text{dom } \alpha_p$ is recorded with the call; we write $\text{par}(q) = (p, \beta)$ and call β the *father* of q . Cancellation always cancels a

procedure and all its descendants. A cancelled procedure is never reactivated, and its historical data and enumerated axioms are retained.

A pair (p, β) is *correct at stage s* if p is active, $\beta \in \text{dom } \alpha_p$, $\beta \preceq D_s$, and $\alpha_p(\beta) \preceq A_s$. The pair order is the transitive closure of

$$(p, \beta) < (p, \gamma) \quad (\beta \prec \gamma), \quad \text{par}(q) < (q, \rho_q).$$

We do not assume in the definition that the correct pairs are linearly ordered; this will follow from the father property.

Every nonroot node δ added to $\text{dom } \alpha_p$ is supplied with a *certificate*. It consists of an integer $k_p(\delta) > |\delta|$, a side $j_p(\delta) < 2$, and strings

$$\zeta_{p,\delta}(\gamma) \quad \text{for all } \gamma \in 2^{k_p(\delta)} \text{ with } \delta \preceq \gamma.$$

The explicit subscript δ prevents any ambiguity between certificates attached to different nodes.

Definition 3.1 (Admissible certificate rule). Consider a surviving procedure p with current child q , and let β be the longest correct node of p before any action at stage s . Put

$$\delta = D_s \upharpoonright (|\beta| + 1), \quad \theta = A_s \upharpoonright (|\beta| + 1).$$

When $\delta \notin \text{dom } \alpha_p$, a *structurally admissible certificate rule* may, by a finite effective test, propose a side $j < 2$, a large $k > |\delta|$, and long strings $\zeta(\gamma)$, ending in 1, for all $\delta \preceq \gamma \in 2^k$, subject to

$$(7) \quad \nexists (\sigma, \zeta) \in \text{dom } \Gamma_{j,s-1} [\hat{\theta} \preceq \sigma \ \& \ \zeta \preceq \eta_{q,j}]$$

and

$$(8) \quad \eta_{q,j} \prec \zeta(\gamma) \quad (\delta \preceq \gamma \in 2^k).$$

The rule is allowed to return no proposal. It must terminate at each stage. No diagonalization property is part of structural admissibility.

Here and below, comparisons with A_s, D_s are effective, because only finitely many bits are involved. If the existence of a required longest node or of a string evaluation has not yet been established, the instructions are understood subject to the inductive verification in Section 4. Safety itself is tested from a finite set of old axioms: if neither side is safe, there is simply no proposal.

3.1. Stage instructions. At stage 0 create p_0 , with label 0, all its strings empty, and $\alpha_{p_0}(\langle \rangle) = \langle \rangle$. Enumerate the empty axioms into each Ξ_j and Γ_j , and record the first budget for its root. For $s > 0$, perform the following steps.

Step 1: cancellation. Cancel every active p for which $\mu_p \not\leq m_s$, together with its descendants. Update correctness records and period budgets for the surviving pairs.

Step 2: certification. For every surviving procedure p which has a child, form β, δ, θ as in Definition 3.1. If δ is not already in the domain and the rule proposes an admissible certificate, say that p *requires attention*. If any procedure requires attention, let p be the strongest such procedure. Record its certificate, set $\alpha_p(\delta) = \theta$, and record a period budget for this new correct node. Cancel all procedures weaker than p and end the stage. No Ξ - or Γ -axioms are enumerated in this case.

Step 3: a new call. If no procedure requires attention, let p be the weakest surviving procedure. Let β be its longest correct node, and put $\delta = D_s \upharpoonright (|\beta| + 1)$. If $\delta \notin \text{dom } \alpha_p$, let k be the current frozen period budget of (p, β) . Otherwise let $k = k_p(\delta)$. Call a new child q with

$$(9) \quad \begin{aligned} r_q &= r_p + 1, & \text{par}(q) &= (p, \beta), & \mu_q &= m_s \upharpoonright k, \\ \rho_q &= D_s \upharpoonright k, & a_q &= \alpha_q(\rho_q) = A_s \upharpoonright k. \end{aligned}$$

In an *ordinary call*, namely when δ is undefined, let $\eta_{q,j}$ be a free extension of $\xi_{j,s,k}^*$ for both $j < 2$. In a *certified call*, let $j_* = j_p(\delta)$, let η_{q,j_*} be a free extension of $\zeta_{p,\delta}(\rho_q)$, and let $\eta_{q,1-j_*}$ be a free extension of $\xi_{1-j_*,s,|\beta|}^*$. For both $j < 2$, enumerate

$$(10) \quad \mu_q \mapsto \eta_{q,j} \in \Xi_j, \quad (a_q, \eta_{q,j}) \mapsto \rho_q \in \Gamma_j.$$

Finally record the root's period budget and complete the historical bound for the stage. Budgets for existing uninterrupted periods are not changed.

4. STRUCTURAL TRANSFER

The purpose of this section is to verify that the preceding protocol works for every structurally admissible certificate rule. The result below separates the reusable structural part of the construction from the particular diagonalization carried out in Section 5.

Theorem 4.1 (Structural transfer). *For every structurally admissible finite effective certificate rule, the protocol in Section 3 has the following properties.*

- (i) *It is well defined at every stage. Each Γ_j is consistent, each $\xi_{j,s}$ is a finite string, and for every active p ,*

$$(11) \quad \xi_{j,s,|\rho_p|} = \eta_{p,j} \quad (j < 2).$$

- (ii) *For every s and $\ell > 0$, some $j < 2$ satisfies*

$$(12) \quad \nexists(\sigma, \zeta) \in \text{dom } \Gamma_{j,s} [(\widehat{A_s \upharpoonright \ell}) \preceq \sigma \ \& \ \zeta \preceq \xi_{j,s}].$$

- (iii) *If $A \not\leq_T D$, the permanent procedures form an infinite chain, and the sets $X_j = \Xi_j(m)$ are total and satisfy*

$$X_j \leq_T m \quad \text{and} \quad \Gamma_j(A, X_j) = D \quad (j < 2).$$

We prove the theorem in several steps. These steps isolate the structural verification of [DGLM13, Section 6]; none uses a property of a diagonalization functional.

4.1. Active records and applicable outputs. We first prove well-definedness and (11) by a simultaneous induction on the stage. In this induction we also prove the elementary ancestry facts and the historical extension facts below. At stage s , all assertions at earlier stages are available. By (6), the old evaluations on m_s are already strings before Step 3 is performed. The verification of the ancestry facts does not use the new candidate strings chosen in Step 3.

Lemma 4.2 (Ancestry and correctness). *At every stage the root procedure survives. If $\text{par}(q) = (p, \beta)$, then*

$$|\beta| < |\rho_q|, \quad \beta \prec \rho_q, \quad \alpha_p(\beta) \prec a_q, \quad \mu_p \prec \mu_q.$$

The maps α_p preserve proper extension on their domains. Every active root is correct. More generally, if a historical procedure q satisfies $\mu_q \preceq m_s$, then $\rho_q \preceq D_s$ and $a_q \preceq A_s$. While q is active, its father is the longest correct node of its parent. Consequently, the correct pairs form a linearly ordered initial segment of the pair order on active procedures.

Proof. The root's modulus string is empty. A new child has root length greater than its father's length: in an ordinary call this follows from the budget rule; in a certified call it follows from $k_p(\delta) > |\delta| > |\beta|$. Its root and its father's node are prefixes of the same D_s , and their labels are prefixes of the same A_s . A new label on a one-bit successor is likewise chosen to extend the predecessor's label. These facts give the required prefix relations and, along ancestry, the proper extension of modulus strings.

A modulus string that still applies at stage s has applied at every stage since its creation, by nonreturn. Equation (4) then shows that the corresponding A - and D -prefixes have not changed. This proves the assertion about historical roots and labels, and in particular about

active roots. Correctness is downward closed in the pair order because both nodes and labels extend along that order.

At its child's creation the father is longest correct by definition. Suppose it first fails to be longest while the child survives. If a new correct successor of the father is added, Step 2 cancels the child. Otherwise an already defined successor has just become correct. The approximations have then changed below its length, which is at most the child's root length. Step 1 cancels the child, again a contradiction. The same argument applies before an action at the next stage when the child survives Step 1.

Within each individual tree, correct nodes are a downward-closed chain of prefixes of D_s . For two different active procedures, the father property says that every correct node of the ancestor is at or below the father leading to the descendant. Iterating along the active chain therefore compares the two pairs. This proves global linearity, as well as the asserted initial-segment property. $\square$

If an old procedure p survives at stage s , the previous-stage identity gives

$$(13) \quad \xi_{j,s,|\rho_p|}^* = \eta_{p,j}.$$

Indeed $m_s \upharpoonright |\rho_p| = m_{s-1} \upharpoonright |\rho_p| = \mu_p$. It follows that a new ordinary child extends its parent's candidate strings. The same holds in a certified call: the certificate extends the candidate of the child that was active at its registration, and that old candidate extended the parent's candidate by induction. Thus, at the current stage,

$$(14) \quad p < q \text{ active} \implies \eta_{p,j} \prec \eta_{q,j} \quad (j < 2).$$

The remaining issue for Ξ_j is the contribution of cancelled procedures whose modulus strings still apply.

Lemma 4.3 (Persistence of an applicable historical modulus string). *Suppose that q was active at a stage $t < s$, is cancelled by the end of stage s , and $\mu_q \preceq m_s$. Put $N = |\rho_q|$. Then*

$$(15) \quad \xi_{j,s,N}^* = \Xi_{j,t}(\mu_q) = \eta_{q,j} \quad (j < 2).$$

If a new procedure p is called at stage s , then

$$(16) \quad |\rho_p| > N \quad \text{and} \quad \eta_{q,j} \prec \eta_{p,j} \quad (j < 2).$$

Proof. All the assertions are included in the stage induction. Let r be the weakest procedure active at both t and s . It exists because the root survives, and it is a proper ancestor of q . Let v be the child of r active at t , and let β be its father. The procedure v is cancelled at some stage $u \in (t, s]$. Since $\mu_v \preceq \mu_q \preceq m_s$, the modulus string μ_v remains applicable throughout this interval.

The procedure r also survives throughout. Thus v is cancelled by an action of r at Step 2 of stage u , not by Step 1 or by an ancestor of r .

The father property identifies the new node as $\beta^+ = D_u \upharpoonright (|\beta| + 1)$. It is correct at u and remains correct through s : its length is at most N , and the approximations below N have not changed since t . Any child of r active at s therefore has father extending β^+ . Such a father was added to $\text{dom } \alpha_r$ at or after stage u .

Suppose a procedure p is called at stage s . If its parent is r , its father δ extends β^+ . In an ordinary call, the budget of its present correct period was chosen no earlier than the addition of δ , so it exceeds the already recorded length N . In a certified call, the certificate of the successor of δ was registered still later, so its large length also exceeds N . If p is further below r , the current child of r was created after u and before s . By the stage induction its root length exceeds N ; so does the root length of p . This proves the length assertion in (16).

For every intervening stage $w \in (t, s)$, if q is still active then a new call at w is below q and has a modulus string of length greater than N . If q is already cancelled, the same statement follows from the length assertion at stage w by induction, since $\mu_q \preceq m_w$. Consequently

no axiom $\mu \mapsto \eta$ with $|\mu| \leq N$ is enumerated between stages t and $s - 1$. Moreover, $m_t \upharpoonright N = m_s \upharpoonright N = \mu_q$. Hence

$$\xi_{j,s,N}^* = \Xi_{j,t}(\mu_q).$$

At stage t , the active-root identity identifies this last string with $\eta_{q,j}$, proving (15). This is an equality of historical evaluations, not merely a comparison of candidate strings.

It remains to verify the candidate extension when p is called at s . If p is below a child of r created before s , use the extension assertion at that earlier creation stage and (14). We may therefore assume that the parent of p is r , with father δ added no earlier than u .

For an ordinary call, the base is $\xi_{j,s,|\rho_p|}^*$, which extends $\xi_{j,s,N}^* = \eta_{q,j}$. For a certified call, put $\delta^+ = D_s \upharpoonright (|\delta| + 1)$ and let $w < s$ be the stage at which δ^+ was certified. Since δ was added at or after u , we have $u < w < s$. The node δ^+ was correct at w but is not correct at s , while all prefixes of length N remained fixed. Therefore $|\delta^+| > N$, so

$$(17) \quad |\delta| \geq N.$$

The side using a free extension of $\xi_{j,s,|\delta|}^*$ thus extends $\eta_{q,j}$ by (15). For the certificate side, let o be the child of r at stage w before certification. It survives Step 1, and $|\rho_o| > |\delta| \geq N$. By the historical equality at stage w and the active-root identity at stage $w - 1$,

$$\eta_{q,j} = \xi_{j,w,N}^* \preceq \xi_{j,w,|\rho_o|}^* = \eta_{o,j}.$$

Every certificate string on that side extends $\eta_{o,j}$. The free extension used for p therefore properly extends $\eta_{q,j}$ as well. This completes the simultaneous proof. $\square$

Proposition 4.4. *At every stage, $\xi_{j,s}$ is a finite string, and (11) holds for every active procedure.*

Proof. If a new procedure p is called, every old modulus string applying to m_s comes from either a surviving ancestor or a cancelled historical procedure. Its associated candidate string is properly extended by $\eta_{p,j}$, respectively by (14) or Lemma 4.3.

Thus the new evaluation is the string $\eta_{p,j}$. If there is no call, no new axiom is enumerated, and consistency follows from (6).

For an old surviving q , a new call has root length greater than $|\rho_q|$, so it does not change the evaluation on the finite oracle prefix $m_s \upharpoonright |\rho_q|$. Equation (13) proves the required identity. For a newly called p , its own axiom applies and all old strings supplied by applicable axioms are shorter prefixes of $\eta_{p,j}$, so the identity also holds. This completes the stage induction, and therefore justifies every finite evaluation used by the protocol. $\square$

4.2. Consistency of the coding functionals. We now use a second stage induction. Its assertions are the consistency of $\Gamma_{j,s}$, the comparison property in Lemma 4.6, and the certificate property in Lemma 4.5. The construction is already well defined by Proposition 4.4. At a certification stage we prove the certificate property from the previous consistency assertion; at a call stage we use already registered certificates to prove the comparison property. This order avoids a dependency on future consistency.

The next lemma records exactly what the safety test at a certification stage guarantees about previously enumerated coding axioms. It is the local fact used in the certified-call case of the consistency proof for the Γ_j .

Lemma 4.5 (Old axioms at a certificate). *Suppose that r certifies $\delta = \beta \frown i$ at stage t , with $\alpha_r(\delta) = \theta$ and selected side j . Let v be the child of r just before that action. For each certificate string $z = \zeta_{r,\delta}(\gamma)$, the following statements hold.*

- (i) $\text{par}(v) = (r, \beta)$ and $\theta \preceq a_v$.
- (ii) If $(\sigma, \zeta) \mapsto \nu$ is in $\Gamma_{j,t-1}$, $\zeta \preceq z$, and $\theta \not\preceq \sigma$, then $\zeta \preceq \eta_{v,j}$.
- (iii) If $(\sigma, \zeta) \mapsto \nu$ is in $\Gamma_{j,t-1}$, $\zeta \preceq z$, and σ is comparable with $\hat{\theta}$, then $\nu \preceq \beta$.

Furthermore, any coding axiom at a later stage whose second component is a prefix of z was already present at stage $t - 1$.

Proof. The father property gives $\text{par}(v) = (r, \beta)$. The child survives Step 1, so $a_v \preceq A_t$. Since $|\rho_v| > |\beta|$ and $\theta = A_t \upharpoonright (|\beta| + 1)$, we have $\theta \preceq a_v$.

For (ii), the strings ζ and $\eta_{v,j}$ are comparable, since both are prefixes of z . Suppose that $\eta_{v,j} \prec \zeta$. Write $\sigma = a_q$, $\zeta = \eta_{q,j}$, and $\nu = \rho_q$ for the procedure responsible for the old axiom. Freshness of candidate lengths implies that q was called after v . Since v remained active up to the certification, q was a descendant of v when it was called. The ancestry relations give $a_v \prec a_q$, contrary to $\theta \not\prec \sigma$.

For (iii), a string comparable with $\hat{\theta}$ cannot extend θ . Part (ii) therefore gives $\zeta \preceq \eta_{v,j}$. The safety condition excludes $\hat{\theta} \preceq \sigma$, so

$$\sigma \prec \hat{\theta}, \quad \text{hence} \quad \sigma \preceq \alpha_r(\beta).$$

The axiom in question is compatible with the old child axiom $(a_v, \eta_{v,j}) \mapsto \rho_v$. By the inductive consistency of $\Gamma_{j,t-1}$, its output ν is compatible with ρ_v . Every coding axiom has equal first-component and output lengths. Thus

$$|\nu| = |\sigma| \leq |\beta|, \quad \nu \preceq \rho_v \upharpoonright |\beta| = \beta.$$

This is the required short-output conclusion.

Finally, no call occurs at stage t , and a candidate created after t is longer than z . It cannot be the second component of an axiom whose second component is a prefix of z . $\square$

Lemma 4.6 (Comparison of coding axioms). *For any historical procedures p, q and any $j < 2$,*

$$(18) \quad \eta_{q,j} \preceq \eta_{p,j} \implies a_q \perp a_p \text{ or } \rho_q \preceq \rho_p.$$

Consequently Γ_j is a consistent Turing functional.

Proof. Consider a new procedure p at stage s . For $q \neq p$ with $\eta_{q,j} \preceq \eta_{p,j}$, freshness implies that q was called before s . We first record the following consequence of the inductive comparison property:

$$(19) \quad \eta_{q,j} \preceq \xi_{j,s,n}^* \implies a_q \perp (A_s \upharpoonright n) \text{ or } \rho_q \preceq D_s \upharpoonright n.$$

To prove it, choose a historical procedure w whose modulus string applies to $m_s \upharpoonright n$ and whose candidate $\eta_{w,j}$ is the longest applicable Ξ_j -string there. Such a w exists, including the root when this string is empty. Then $|\rho_w| \leq n$ and $\eta_{q,j} \preceq \eta_{w,j}$. By the comparison property at earlier stages, either $a_q \perp a_w$ or $\rho_q \preceq \rho_w$. Since $\mu_w \preceq m_s$, we have $a_w \preceq A_s \upharpoonright n$ and $\rho_w \preceq D_s \upharpoonright n$. This proves (19).

If $\eta_{p,j}$ is a free extension of $\xi_{j,s,n}^*$, where n is its root length or its father's length, Lemma 2.2 reduces $\eta_{q,j} \preceq \eta_{p,j}$ to $\eta_{q,j} \preceq \xi_{j,s,n}^*$. The first case of (19) gives incompatibility with a_p , and the second gives $\rho_q \preceq \rho_p$.

Otherwise p is a certified call on its parent's selected side. Write its parent as r , its father as β , and $\delta = D_s \upharpoonright (|\beta| + 1)$. Let $t < s$ be the certificate stage and $\theta = \alpha_r(\delta)$. By maximal correctness of the father, δ is not correct now. Its predecessor is correct and its node is a prefix of D_s . Therefore

$$(20) \quad \hat{\theta} \preceq a_p.$$

By the free-extension lemma, $\eta_{q,j}$ is a prefix of the relevant certificate string. Its coding axiom was already present before t , by the last assertion of Lemma 4.5. If a_q is compatible with a_p , it is comparable with $\hat{\theta}$. Part (iii) of that lemma yields $\rho_q \preceq \beta \prec \rho_p$. This proves (18) for the new call. At stages without a call, there is no new pair of coding axioms to check. Together with Lemma 4.5, this completes the second induction.

Finally, two axioms can apply to a common pair of oracles only if their first components are compatible and their second components are comparable. Apply (18) in the direction of the shorter second component. Their outputs are comparable, which is exactly consistency. $\square$

Corollary 4.7 (No deeper old axiom). *For every historical p and $j < 2$, there is no coding axiom $(\sigma, \zeta) \mapsto \nu$ in Γ_j such that $a_p \prec \sigma$ and $\zeta \preceq \eta_{p,j}$.*

Proof. Write the axiom as $(a_q, \eta_{q,j}) \mapsto \rho_q$. Lemma 4.6 would give $\rho_q \preceq \rho_p$, whereas $a_p \prec a_q$ and equality of first-component and output lengths give $|\rho_p| < |\rho_q|$. $\square$

4.3. The history of a certified call. A certified call may occur long after its certificate was registered. The following lemma identifies what can have happened in the intervening stages and, in particular, shows that no procedure called after the certificate stage can still have an applicable modulus string at the time of the call.

Lemma 4.8. *Suppose p is called at stage s using a certificate registered at stage $t < s$ by its parent r . Write $\text{par}(p) = (r, \beta)$ and $\delta = \beta \frown i$ for the certified node. Then:*

- (i) *The first position at which A_t and A_s differ is $|\beta|$.*
- (ii) *Every old procedure q with $\mu_q \preceq m_s$ was called before stage t . Consequently, $\xi_{j,s}^* \preceq \xi_{j,t}$ for both sides.*
- (iii) *If an axiom in $\Gamma_{j,s-1}$ has its second component contained in $\xi_{j,t}$ or in any certificate string registered at t , then that axiom already belongs to $\Gamma_{j,t}$.*

Proof. The label of β is a prefix of both A_t and A_s . At stage t , the label of δ extends it correctly by one bit. At stage s , the node δ is a prefix of D_s but its label is incorrect, since β is longest correct. This proves (i).

Here is a direct proof of (ii) using the persistence mechanism. Suppose that an old q was called after t and that $\mu_q \preceq m_s$. The procedure r survives from t to s and cancels all its descendants at t . Thus q was called below r . Let v be the immediate child of r on the ancestry of q . It was called after t , and $\mu_v \preceq \mu_q \preceq m_s$. It is no longer active when p is called. Because its modulus string has never ceased to apply and r survives, its cancellation must be a Step 2 action of r at some stage $u \in (t, s)$.

Let λ be the father of v . At stage u , r adds the correct node λ^+ extending λ by one bit. Since $|\lambda^+| \leq |\rho_v|$, this node and its label remain correct through s . Therefore the longest correct node β at stage s extends λ^+ . But β already belonged to $\text{dom } \alpha_r$ before stage t , while λ^+ was first added after t . This contradicts the tree property of the domain. No call occurs at t itself. Hence all old applicable procedures were called before t .

Their modulus strings also applied at stage t , by nonreturn. Taking the union of their candidate strings proves $\xi_{j,s}^* \preceq \xi_{j,t}$. Part (iii) follows from freshness: a candidate chosen after t is longer than every string mentioned in (iii), so cannot be their prefix. $\square$

4.4. An available side at every stage. The central safety invariant is that, for every relevant flipped A_s -prefix, at least one of the two sides remains free of a conflicting old coding axiom. This is the mechanism that guarantees the existence of a safe side whenever a procedure is ready to act.

Lemma 4.9 (Available-side invariant). *For every stage s and every $\ell > 0$, some $j < 2$ satisfies (12).*

Proof. Induct on s . The assertion holds at stage 0. At stage s , a new coding axiom has first component a prefix of A_s , so it cannot extend $(\widehat{A_s} \upharpoonright \ell)$. It is therefore enough to exclude old coding axioms.

If there is no call at s , or the call is ordinary, put $t = s - 1$. If a certified call is made, let t be its certificate-registration stage. Let n be the length of the longest common initial segment

of A_t and A_s , allowing $n = \infty$ when they are equal. For a certified call, Lemma 4.8(i) says that n is its father's length.

First suppose $\ell \leq n$. At stage t choose a side j that satisfies the invariant for ℓ . The flipped prefixes at stages t and s agree. Suppose, toward a contradiction, that an old axiom $(\sigma, \zeta) \mapsto \nu$ has

$$(\widehat{A_s \upharpoonright \ell}) \preceq \sigma, \quad \zeta \preceq \xi_{j,s}.$$

With no call, $\xi_{j,s} = \xi_{j,s}^* \preceq \xi_{j,t}$, contradicting our choice at t . With an ordinary call, the free-extension lemma reduces ζ to a prefix of the old base, hence of $\xi_{j,t}$; again we have a contradiction.

For a certified call, first consider the side that uses a free extension of the old base truncated at the father's length. The free-extension lemma and Lemma 4.8(ii) reduce ζ to a prefix of $\xi_{j,t}$; part (iii) of that lemma puts its axiom at stage t . This contradicts the invariant at t . On the selected side, the free-extension lemma instead puts ζ below a certificate string. Its axiom was already present at t . Let θ be the label registered with that certificate, and let v be the child present at its registration. Since $\ell \leq n$ and σ extends the flipped prefix at length ℓ , it cannot extend θ , which is a prefix of A_t of length $n + 1$. Lemma 4.5(ii) gives $\zeta \preceq \eta_{v,j} \preceq \xi_{j,t}$. The last inclusion holds even though v was cancelled at t : its modulus string still applied at that certification stage and no new Ξ -axiom was enumerated there. This is the remaining contradiction for $\ell \leq n$.

Now suppose $\ell > n$. Choose a side j on which $\xi_{j,s}$ is either $\xi_{j,s}^*$ or a free extension of an initial segment of $\xi_{j,s}^*$. Both sides qualify with no call or an ordinary call; in a certified call, use the side not selected by the certificate. Any old second component $\zeta \preceq \xi_{j,s}$ is consequently a prefix of $\xi_{j,s}^*$.

Choose an old applicable procedure q such that $\eta_{q,j} = \xi_{j,s}^*$, using the root if this string is empty. In the first two cases q was called by $t = s - 1$; in the certified case it was called before t by Lemma 4.8(ii). Since its modulus string applies at both t and s , its root and label are prefixes of both approximations. In particular $|\rho_q| \leq n$, as A_t and A_s disagree at n . Thus

$$a_q \preceq A_s \upharpoonright n \prec (\widehat{A_s \upharpoonright \ell}).$$

An old axiom whose first component extends this flipped prefix and whose second component is contained in $\xi_{j,s}^* = \eta_{q,j}$ would violate Corollary 4.7. This proves the invariant. $\square$

Corollary 4.10 (Safety before action). *Suppose a child q is active at the end of stage $s - 1$ and survives Step 1 of stage s . Let $\text{par}(q) = (p, \beta)$ and $\theta = A_s \upharpoonright (|\beta| + 1)$. At least one side satisfies the safety condition (7) with respect to the old axioms.*

Proof. Put $\ell = |\beta| + 1 \leq |\rho_q|$. Survival implies $A_s \upharpoonright \ell = A_{s-1} \upharpoonright \ell$. Apply Lemma 4.9 at stage $s - 1$, not at stage s . For both sides $\eta_{q,j} \preceq \xi_{j,s-1}$, so the side it supplies is safe for the smaller candidate prefix as well. $\square$

4.5. Permanent procedures and coding at the limit. Assume now that $A \not\leq_T D$. Call a procedure *permanent* if it is never cancelled. For such a procedure p , a node β is *truly correct* if $\beta \preceq D$ and $\alpha_p(\beta) \preceq A$. This is equivalent to the pair being correct at all sufficiently late stages, and also to its being correct at infinitely many stages.

Lemma 4.11. *Every permanent procedure has a longest truly correct node, requires attention at only finitely many stages, and has a permanent child.*

Proof. The root of p is truly correct, because μ_p applies at every later stage. If truly correct nodes were unbounded in length, the tree and label-extension properties would imply that every defined node $\nu \preceq D$ has its correct A -label: choose a longer truly correct node and use preservation of initial segments. With oracle D , given n , enumerate α_p until a node $\nu \in \text{dom } \alpha_p$ with $\nu \preceq D$ and $|\nu| > n$ is found, and output $\alpha_p(\nu)(n)$. The search terminates by

unboundedness and is correct by the preceding observation. This computes A from D , with the finite index of p and its root built into the reduction.

The hypothesis excludes this. Let β be the longest truly correct node and put $\delta = D \upharpoonright (|\beta| + 1)$.

Choose s_0 after β has been defined and after the approximations below $|\delta|$ have settled. If δ is ever defined, also choose s_0 after its definition. At all stages beyond s_0 , the longest correct node of p is β . If δ is defined, Step 2 does not request another certificate for it. If δ is never defined, a late request would lead to its definition: no stronger procedure can act, since that would cancel the permanent p . This would contradict our choice of δ as never defined. Thus p no longer requires attention.

After s_0 , whenever p has no child following Step 1, it is the weakest active procedure and calls a child in Step 3. The possible root lengths of these new children are bounded. They use either the fixed certificate length for δ , or the fixed period budget of β , whose correctness no longer breaks. Choose a later stage after m_s has settled below this bound. A child created after that stage cannot be cancelled by Step 1. It cannot be cancelled by p , which no longer requires attention, or by an ancestor, since p survives. Hence it is permanent. If no new child is created after that stage, the already present child is itself permanent. $\square$

Completion of Theorem 4.1. Proposition 4.4 and Lemmas 4.6 and 4.9 establish the first two assertions. Starting from the permanent root and repeatedly applying Lemma 4.11, choose permanent children $p_0, p_1, \dots$, with $r_{p_r} = r$. On each side $j < 2$, their candidate strings $\eta_{p_r, j}$ strictly increase with r , and every corresponding modulus string is a prefix of m .

Any two Ξ_j -axioms whose left-hand strings are prefixes of m are both present and applicable at a sufficiently late stage. Proposition 4.4 then makes their outputs comparable. Thus $\Xi_j(m)$ is a consistent union, and the permanent candidates make it infinite. To compute its n th bit with oracle m , enumerate axioms until one is found whose left-hand string is a prefix of m and whose output has length greater than n . Totality guarantees termination and the established consistency guarantees correctness. This proves $X_j = \Xi_j(m) \leq_T m$, without asserting consistency of Ξ_j on unrelated oracles.

Finally, $\rho_{p_r} \preceq D$, $a_{p_r} \preceq A$, and $\eta_{p_r, j} \preceq X_j$. The corresponding coding axioms therefore apply to (A, X_j) . Their output lengths strictly increase, so they cover D . Consistency of Γ_j implies $\Gamma_j(A, X_j) = D$, as required. $\square$

5. AVOIDING A DESIGNATED COLUMN

We now specialize the structural protocol. The key point is that the coding functionals will continue to recover the whole target D , while the certificate rule diagonalizes only against one designated column of D . This separation is what allows the coded set and the avoided set to play different roles.

For an infinite binary sequence D , define its odd column by $\pi(D)(n) = D(2n + 1)$. For a finite binary string γ , $\pi(\gamma)$ has the same definition on those n with $2n + 1 < |\gamma|$. This is a literal finite projection, including on incorrect approximations. It is not an application of a possibly divergent Turing reduction to an arbitrary finite oracle.

Lemma 5.1 (Column avoidance). *Let C be c.e., let $A, D \leq_T C$, and suppose that $A \not\leq_T D$. If $E = \pi(D)$ is noncomputable, there are sets $X_0, X_1 \leq_T C$ such that*

$$D \leq_T A \oplus X_j \quad (j < 2), \quad \neg(E \leq_T X_0 \ \& \ E \leq_T X_1).$$

Proof. Choose approximations and the modulus as in Section 2, so $m \leq_T C$. Effectively enumerate all ordered pairs of binary-valued Turing functionals as $(\Phi_{r,0}, \Phi_{r,1})$, for $r \in \omega$. We meet

$$(21) \quad \mathcal{R}_r : \quad \Phi_{r,0}^{X_0} \neq E \quad \text{or} \quad \Phi_{r,1}^{X_1} \neq E.$$

The enumeration includes pairs with different indices on the two sides. If $\Phi_{r,j}^{X_j}$ is partial, then it is of course unequal to the total characteristic function of E .

We specify the certificate rule for a procedure p with label r and current child q . At stage s , on each side inspect all witnesses, with codes and computation times at most s , to a splitting above $\eta_{q,j}$. Such a witness consists of finite strings $\tau^0, \tau^1 \succ \eta_{q,j}$ and an argument n with

$$(22) \quad \Phi_{r,j}^{\tau^0}(n) \downarrow = 0, \quad \Phi_{r,j}^{\tau^1}(n) \downarrow = 1.$$

The computations in (22) use only positions within the specified finite strings. The bounded tests are exhaustive as s increases for any fixed candidate base.

If a splitting is found on a safe side j , select one by a fixed effective order. Choose a fresh large k such that

$$k > |\delta| \quad \text{and} \quad k > 2n + 1,$$

where $\delta = D_s \upharpoonright (|\beta| + 1)$ is the node to be certified. For every $\delta \preceq \gamma \in 2^k$, choose a sufficiently long extension $\zeta(\gamma)$, ending in 1, of

$$\tau^{1-\gamma(2n+1)}.$$

All these extensions can be chosen effectively after the bounded splitting test. Their lengths need not be at most s ; the action is finite, not time-bounded by the stage number. They satisfy

$$(23) \quad \Phi_{r,j}^{\zeta(\gamma)}(n) = 1 - \gamma(2n + 1).$$

They properly extend $\eta_{q,j}$ and obey the freshness convention. If neither safe side has a splitting observed by that stage, return no proposal. This is a finite effective structurally admissible rule. For this application, whenever such a proposal is acted upon at Step 2, we also retain its splitting argument n with the registered certificate. This is auxiliary data for the present verification and plays no role in the structural protocol.

Theorem 4.1 now gives total sets $X_j \leq_T m \leq_T C$ and $\Gamma_j(A, X_j) = D$ for both j . Fix r , let p be the permanent procedure with label r , and let β be its longest truly correct node. Put $\delta = D \upharpoonright (|\beta| + 1)$, and let q be the permanent child of p . Its father is β : the father property holds at all late stages, where the longest correct node has stabilized.

Suppose first that $\delta \in \text{dom } \alpha_p$. Its certificate must have been registered before q was called, since a later registration would cancel q . Write its data as j, k, n and $\zeta_{p,\delta}(\gamma)$. The call of q is a certified call, with

$$\rho_q = D \upharpoonright k, \quad \zeta_{p,\delta}(\rho_q) \prec \eta_{q,j} \preceq X_j.$$

Here we use the *long root* ρ_q , not just the one-bit successor δ . Since $k > 2n + 1$, equation (23) gives

$$(24) \quad \Phi_{r,j}^{X_j}(n) = 1 - \rho_q(2n + 1) = 1 - E(n).$$

Thus $\mathcal{R}_r$ is met.

Suppose next that $\delta \notin \text{dom } \alpha_p$, and toward a contradiction assume

$$(25) \quad \Phi_{r,0}^{X_0} = E = \Phi_{r,1}^{X_1}.$$

There must be a splitting above $\eta_{q,j}$ on each side j . Indeed, if there were no splitting on a side, a computable procedure with the finite string $\eta_{q,j}$ built into its program could compute $E(n)$ as follows: search for a finite extension of this string on which $\Phi_{r,j}(n)$ converges, and return the resulting bit. Equality (25) supplies such an extension along X_j , so the search terminates. The absence of a splitting makes its answer unique and equal to the value along X_j . This would make E computable, a contradiction.

Choose a sufficiently late stage at which p and q are present, the longest correct node is β , and the bounded searches have found a splitting on *both* sides above the now fixed bases $\eta_{q,j}$. Corollary 4.10 supplies at least one safe side at this stage. The rule therefore makes p require

attention. No stronger procedure can act without cancelling the permanent p , so p acts and defines δ . This contradicts the present case. Hence (25) is impossible, and $\mathcal{R}_r$ is met.

Every requirement is met. If both X_0 and X_1 computed E , the ordered pair of functionals witnessing these two reductions would occur in our enumeration and violate its requirement. Therefore at least one side fails to compute E . $\square$

6. THE CONE-AVOIDANCE CONCLUSION

We finish by reducing the general cone-avoidance statement to the column-avoidance lemma. The only nontrivial case is when the set to be avoided is already computable from B ; we then place it in a separate column of a target Turing-equivalent to B .

Proof of Theorem 1.1. Fix a noncomputable E . If $E \not\leq_T B$, take $X = B$. Then $X \leq_T C$, $B \leq_T A \oplus X$, and $E \not\leq_T X$.

Otherwise $E \leq_T B$. Define the literal join

$$D(2n) = B(n), \quad D(2n+1) = E(n).$$

Then $D = B \oplus E \equiv_T B$, so $A, D \leq_T C$ and $A \not\leq_T D$. The odd column of D is the noncomputable set E . Lemma 5.1 gives two candidates $X_j \leq_T C$ with $D \leq_T A \oplus X_j$, at least one of which does not compute E . For that side let $X = X_j$. Then

$$B \leq_T D \leq_T A \oplus X \quad \text{and} \quad E \not\leq_T X.$$

$\square$

7. CONCLUDING REMARKS

Theorem 1.1 shows that the no-least-join phenomenon extends to cone avoidance for every prescribed nonzero degree in the following precise sense: whenever c is c.e. and $a, b < c$ with $a \not\leq b$, the restricted difference filter

$$\mathcal{F}_{a,b}^c = \{x \leq c : b \leq a \vee x\}$$

is contained in no nonzero Turing cone. Equivalently, $\mathbf{0}$ is its only common lower bound. Thus Question 1.19 of [DGLM13] has a negative answer.

This also sharpens the interpretation of the dependence phenomenon in Theorem 1.18 of [DGLM13]. One restricted difference filter may be contained in another even though the first has no nonzero common lower bound. Hence such a dependence cannot, under the hypotheses considered here, be explained by a fixed nonzero degree computed by every witness to the first cupping requirement.

The hypothesis $a \not\leq b$ is essential in this form. Theorem 1.11 of [DGLM13] gives configurations with $a \leq b$ in which b itself is a nonzero common lower bound. The present result therefore rules out one natural source of additional extension obstructions, but does not provide a general characterization of such obstructions. In particular, it does not settle the sufficiency of the anti-cupping condition or the two-quantifier theory of the Δ_2^0 degrees with join.

APPENDIX A. RELATION WITH THE NO-LEAST-JOIN CONSTRUCTION

The following correspondence identifies the structural ingredients borrowed from [DGLM13, Section 6, pp. 2999–3015]. The references in the right-hand column use the numbering of the published version of [DGLM13]. The present proof is given in full, so none of these correspondences is used in place of a proof of a transfer claim.

Structural correspondence with the DGLM no-least-join construction

Present argument	Corresponding ingredient in DGLM
Lemma 2.1 and Section 2.1	Sections 6.2.1 and 6.2.5; Step 3 of Section 6.3
Lemma 4.2	Results 6.10–6.18
Lemma 4.3 and Proposition 4.4	Lemmas 6.8–6.9, 6.19–6.20; Proposition 6.7
Lemmas 4.5 and 4.6	Lemmas 6.21–6.24
Lemma 4.8 and Corollary 4.7	Lemmas 6.27–6.28
Lemma 4.9 and Corollary 4.10	Lemma 6.25 and Corollary 6.26
Lemma 4.11 and the limit coding argument	Lemma 6.29, Claims 6.30–6.32, and Section 6.4.9
Lemma 5.1	Column-avoidance modification of the certificate rule and final verification

Three formulation choices are worth recording. First, the minimum in (3) is the beginning of the current stable interval; taking the greatest stage in that interval would give s and would not yield the required approximation. Second, the safety corollary is explicitly derived from the invariant at the *previous* stage. Third, final diagonalization uses the long root whose length was chosen with the certificate. The coding functionals retain the full target D and keep the first-component and output lengths equal throughout.

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