

A STRENGTHENED FORM OF DĚGTEV'S ARGUMENT AND D -MAXIMAL MANY-ONE DEGREES

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ABSTRACT. Děgtev proved that if a noncomputable computably enumerable set U satisfies his property (R) and the many-one degree of U contains no simple set, then the 1-degree of U is minimal. We show that his proof yields the stronger conclusion that, for every noncomputable $A \leq_m U$,

$$U \leq_1 A.$$

In particular, the injectivity of the reduction $A \leq_1 U$ with which Děgtev begins his proof is not needed. Using the Cholak–Gerdes–Lange classification, we show that for a noncomputable D -maximal set D , the hypothesis that its many-one degree contains no simple set is equivalent to D being of one of Types 3–10. Hence, for every such D and every noncomputable $A \leq_m D$, one has $D \leq_1 A$. Together with Maslova's theorem for the simple-set cases, this recovers as a corollary the existence of a least finite-one degree in every noncomputable c.e. many-one degree containing a D -maximal set.

1. INTRODUCTION

Richter, Stephan, and Zhang asked whether every noncomputable many-one degree has a least finite-one degree inside it [5, p. 17]. The question has a negative answer even among computably enumerable (c.e.) many-one degrees [1]. A first version of the present preprint established a positive answer for the subclass of c.e. many-one degrees containing a D -maximal set, using finite-one arguments adapted to the different cases in the Cholak–Gerdes–Lange classification.

After that version was posted, V. Yu. Shavrukov pointed out in correspondence a stronger one-one statement for Types 3–10, drew attention to Děgtev's 1976 paper [3], and suggested that the stronger conclusion is essentially implicit in Děgtev's proof. Děgtev's Assertion 1 [3, p. 751] states that if a noncomputable c.e. set U satisfies property (R) and its many-one degree contains no simple set, then the 1-degree of U is minimal. His proof starts with a noncomputable set $A \leq_1 U$ and constructs a one-one reduction of U to A .

The purpose of the present revision is to verify Shavrukov's suggestion in detail: the same construction works when the initial hypothesis $A \leq_1 U$ is weakened to $A \leq_m U$.

Theorem 1.1. *Let U be a noncomputable c.e. set satisfying property (R) , and suppose that the many-one degree of U contains no simple set. If A is noncomputable and*

$$A \leq_m U,$$

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then

$$U \leq_1 A.$$

We give a complete modernized verification, including the step $A \equiv_m U$ used at the beginning of D egtev's proof and the verification that the final reduction remains injective when the initial reduction is not assumed to be one-one.

For D -maximal sets, property (R) is the superset formulation of D -maximality. Using the Cholak–Gerdes–Lange classification, we show that the many-one degree of a D -maximal set contains a simple set if and only if the set is of Type 1 or Type 2. Thus Theorem 1.1 yields a uniform conclusion for all the remaining types.

Corollary 1.2. *Let D be a noncomputable D -maximal set of one of Types 3–10. If A is noncomputable and $A \leq_m D$, then*

$$D \leq_1 A.$$

For Types 1–2, Maslova's theorem on simple sets supplies the finite-one conclusion. Consequently, the result proved independently in the first version follows from Maslova's theorem, the strengthened reading of D egtev's argument established here, and the CGL classification.

Corollary 1.3. *Every noncomputable c.e. many-one degree containing a D -maximal set contains a least finite-one degree.*

It would be interesting to determine for which further natural classes of many-one degrees the existence of a least finite-one degree still holds.

2. PRELIMINARIES

All sets are subsets of ω . For sets $A, B \subseteq \omega$, we write $A \leq_m B$ if there is a total computable function f such that

$$x \in A \iff f(x) \in B$$

for every x . If f can be chosen injective, we write $A \leq_1 B$. We write $A \leq_{fo} B$ if f can be chosen with all fibres finite. Thus

$$A \leq_1 B \implies A \leq_{fo} B \implies A \leq_m B.$$

For $\rho \in \{1, m, fo\}$, write

$$A \equiv_\rho B$$

if

$$A \leq_\rho B \quad \text{and} \quad B \leq_\rho A.$$

The equivalence classes induced by $\equiv_\rho$ are the ρ -degrees.

We say that A represents the least finite-one degree inside its many-one degree if

$$A \leq_{fo} B \quad \text{for every } B \equiv_m A.$$

A c.e. set S is *simple* if $\overline{S}$ is infinite and contains no infinite c.e. subset.

We shall use the following formulation of D egtev's property (R) .

Definition 2.1. A noncomputable c.e. set U satisfies *property (R)* if, for every c.e. set $B \supseteq U$, whenever $B \setminus U$ is not c.e. there exists a computable set R such that

$$U \subseteq R \subseteq B.$$

This is exactly the superset formulation of D -maximality used by Cholak–Gerdes–Lange; see [2, Definition 2.4 and Lemma 2.5]. Throughout this paper, D -maximal sets are understood to be noncomputable.

3. A STRENGTHENED FORM OF DĚGTEV'S ARGUMENT

We first make explicit a fact used at the beginning of DĚgtev's proof.

Lemma 3.1. *Let U be a noncomputable c.e. set satisfying property (R). If A is noncomputable and $A \leq_m U$, then $U \leq_m A$. Hence $A \equiv_m U$.*

Proof. Let f be a many-one reduction of A to U , and put

$$F := \text{rng}(f).$$

Since U is c.e., the set A is c.e. We first claim that $F \setminus U$ is not c.e. Indeed, for every x ,

$$x \notin A \iff f(x) \notin U \iff f(x) \in F \setminus U.$$

Thus

$$\overline{A} = f^{-1}(F \setminus U).$$

If $F \setminus U$ were c.e., then $\overline{A}$ would be c.e., and hence A would be computable, a contradiction.

Apply property (R) to the c.e. set $U \cup F$. There is a computable set C such that

$$U \subseteq C \subseteq U \cup F.$$

Fix $a \in A$ and $b \notin A$, which exist because A is noncomputable. We define a total computable function h . On input x , if $x \notin C$, set $h(x) = b$. If $x \in C$, run in parallel an enumeration of U and the search for a y such that $f(y) = x$. Since $C \subseteq U \cup F$, at least one of these searches terminates. If x enters U first, set $h(x) = a$; if a preimage y is found first, set $h(x) = y$ (ties may be resolved in favour of the first alternative).

If $x \notin C$, then $x \notin U$ and $h(x) = b \notin A$. If $x \in C$ and $h(x) = a$, then $x \in U$ and $h(x) \in A$. Finally, if $h(x) = y$ with $f(y) = x$, then

$$h(x) = y \in A \iff f(y) = x \in U.$$

Therefore

$$x \in U \iff h(x) \in A$$

for all x , so $U \leq_m A$. □

Proof of Theorem 1.1. Fix a total computable many-one reduction f of A to U . By Theorem 3.1, $A \equiv_m U$. Hence A is not simple, since by hypothesis the many-one degree of U contains no simple set.

The set A is infinite and c.e., so choose an infinite computable set

$$A_1 \subseteq A.$$

Since A is noncomputable, $\overline{A}$ is infinite; since A is not simple, $\overline{A}$ contains an infinite c.e. subset, and therefore an infinite computable set

$$A_2 \subseteq \overline{A}.$$

Put $F = \text{rng}(f)$. As in the proof of Theorem 3.1, $F \setminus U$ is not c.e. Applying property (R) to $U \cup F$, choose a computable set A_3 such that

$$U \subseteq A_3 \subseteq U \cup F.$$

The set A_3 is infinite because it contains the noncomputable c.e. set U .

We construct a total computable function g recursively, preserving the following properties after $g(0), \dots, g(n-1)$ have been defined:

- (i) the values $g(0), \dots, g(n-1)$ are pairwise distinct;
- (ii) for every $x < n$,

$$x \in U \iff g(x) \in A;$$

- (iii) if $x < n$ and $g(x) \notin A_1 \cup A_2$, then

$$g(x) = \min\{y : f(y) = x\}.$$

Suppose g has been defined below n .

If $n \notin A_3$, then $n \notin U$. Let $g(n)$ be the least element of A_2 not among $g(0), \dots, g(n-1)$. This value exists because A_2 is infinite. It is fresh, belongs to $\overline{A}$, and therefore preserves (i)–(iii).

Assume now that $n \in A_3$. Run an enumeration of U in parallel with the sequential search, in increasing order of y , for the least y such that $f(y) = n$. Since

$$A_3 \subseteq U \cup F,$$

at least one of the two searches terminates (ties are resolved in favour of the first alternative).

If n enters U first, let $g(n)$ be the least unused element of A_1 . Then $g(n) \in A$, the value is fresh, and (i)–(iii) are preserved.

Otherwise, suppose that

$$y_0 := \min\{y : f(y) = n\}$$

is found first. There are three cases.

If $y_0 \in A_1$, then $y_0 \in A$, so $n = f(y_0) \in U$. Let $g(n)$ be the least unused element of A_1 .

If $y_0 \in A_2$, then $y_0 \notin A$, so $n = f(y_0) \notin U$. Let $g(n)$ be the least unused element of A_2 .

Finally, suppose $y_0 \notin A_1 \cup A_2$, and set

$$g(n) := y_0.$$

Then

$$g(n) \in A \iff f(g(n)) = n \in U,$$

so (ii) holds, and (iii) holds by definition. It remains only to check that y_0 has not been used before. If $y_0 = g(x)$ for some $x < n$, then $g(x) \notin A_1 \cup A_2$, and the inductive hypothesis (iii) gives

$$f(g(x)) = x.$$

Hence

$$f(y_0) = x.$$

But $f(y_0) = n$, so $x = n$, contradicting $x < n$. Thus the new value is fresh and (i) is also preserved.

This completes the construction. Each stage is effective: membership in A_1, A_2, A_3 is decidable, unused elements of A_1 and A_2 can be found effectively, and the parallel search

terminates because $A_3 \subseteq U \cup F$. Hence g is total computable. By (i), it is injective, and by (ii),

$$x \in U \iff g(x) \in A$$

for every x . Therefore $U \leq_1 A$. $\square$

Remark 3.2 (A slightly more general form). The proof actually uses the hypothesis that the many-one degree of U contains no simple set only to ensure that A is not simple. Thus the same argument proves the slightly more general statement that, if U satisfies property (R), $A \leq_m U$, and A is noncomputable and nonsimple, then $U \leq_1 A$.

Remark 3.3 (On Děgtev's proof). The proof of Assertion 1 [3, pp. 751–752] starts with a noncomputable set $A \leq_1 U$. The preceding proof shows that the injectivity of this initial reduction is not used: a many-one reduction suffices. The English version consulted here states that $f(\mathbb{N}) \setminus U$ is r.e.; however, both the argument and the application of property (R) immediately following it require this set to be *not* r.e., as proved above.

4. TRANSLATION INTO THE CGL CLASSIFICATION

We use the standard facts that, among D -maximal sets, Type 1 is exactly the simple case, and Type 2 is characterized by the existence of an infinite computable set

$$R \subseteq \overline{D}$$

such that $D \sqcup R$ is simple; see [2, Theorem 3.10, Definition 3.11, and Lemma 3.17].

For D -maximal sets we shall also use [2, Theorem 4.1(1),(2)].

Proposition 4.1. *Let D be a D -maximal set. The many-one degree of D contains a simple set if and only if D is of Type 1 or Type 2. Consequently, the D -maximal sets of Types 3–10 are exactly those whose many-one degree contains no simple set.*

Proof. If D is of Type 1, then D itself is simple. Suppose now that D is of Type 2. Choose an infinite computable $R \subseteq \overline{D}$ such that

$$S := D \sqcup R$$

is simple.

Since D is noncomputable, choose $a \in D$, and since S is simple choose $b \notin S$. Define

$$\alpha(x) = \begin{cases} x, & x \notin R, \\ b, & x \in R, \end{cases} \quad \beta(x) = \begin{cases} x, & x \notin R, \\ a, & x \in R. \end{cases}$$

Then α reduces D to S and β reduces S to D , so $D \equiv_m S$. Thus the many-one degree of D contains a simple set.

Conversely, suppose that a simple set S satisfies $S \equiv_m D$. Fix a many-one reduction f of S to D , and put $F = \text{rng}(f)$. As before, $F \setminus D$ is not c.e., since otherwise

$$\overline{S} = f^{-1}(F \setminus D)$$

would be c.e. Since D is D -maximal, property (R) gives a computable set C such that

$$D \subseteq C \subseteq D \cup F.$$

The set $C \setminus D$ is infinite; otherwise D would be a finite variant of the computable set C and hence computable.

We claim that D is simple in C , i.e. that $C \setminus D$ contains no infinite c.e. subset. Suppose, towards a contradiction, that E is an infinite c.e. subset of $C \setminus D$. Since

$$C \setminus D \subseteq F \setminus D,$$

every element of E has an f -preimage. Enumerate E without repetition as $e_0, e_1, \dots$ and, for each i , let

$$y_i := \min\{y : f(y) = e_i\}.$$

Then the y_i are pairwise distinct, form an infinite c.e. set, and satisfy $y_i \notin S$ because $f(y_i) = e_i \notin D$. This contradicts the simplicity of S . Hence D is simple in C .

It follows that D is a maximal subset of C . Indeed, let $D \subseteq W \subseteq C$ be c.e. If $W \setminus D$ is c.e., then simplicity in C makes it finite. Otherwise property (R) gives a computable set H with

$$D \subseteq H \subseteq W.$$

Then the computable set

$$C \setminus H \subseteq C \setminus D$$

is finite, and hence $C \setminus W$ is finite.

Put $R_0 := \overline{C}$. Then R_0 is computable. We claim that $D \cup R_0$ is maximal. Indeed,

$$\overline{D \cup R_0} = C \setminus D$$

is infinite, since D is simple in C . Let W be c.e. with

$$D \cup R_0 \subseteq W.$$

Then

$$D \subseteq W \cap C \subseteq C.$$

Since D is a maximal subset of C , either

$$(W \cap C) \setminus D$$

is finite or

$$C \setminus W$$

is finite. In the first case,

$$W \setminus (D \cup R_0) = (W \cap C) \setminus D$$

is finite, while in the second case

$$\overline{W} = C \setminus W$$

is finite. Hence $D \cup R_0$ is maximal. If R_0 is infinite, [2, Theorem 4.1(2)] implies that D is of Type 2. If R_0 is finite, then D is a finite variant of the maximal set $D \cup R_0$, hence is itself maximal; therefore [2, Theorem 4.1(1)] implies that D is of Type 1. This proves the converse. $\square$

Proof of Theorem 1.2. Every D -maximal set satisfies property (R) by [2, Definition 2.4 and Lemma 2.5]. By Theorem 4.1, if D is of one of Types 3–10, then its many-one degree contains no simple set. The conclusion now follows from Theorem 1.1. $\square$

In particular, the published conclusion of Dęgtev's Assertion 1 is recovered in the following CGL formulation.

Corollary 4.2. *Every D -maximal set of one of Types 3–10 has minimal 1-degree.*

Proof. Let $A \leq_1 D$ be noncomputable. Then $A \leq_m D$, so [Theorem 1.2](#) gives $D \leq_1 A$. Hence $A \equiv_1 D$. $\square$

5. THE FINITE-ONE CONSEQUENCE

We use Maslova's theorem that every noncomputable c.e. simple set represents the least finite-one degree inside its many-one degree [\[4\]](#); see also [\[5, p. 3\]](#).

Proof of [Theorem 1.3](#). Let $\mathbf{d}$ be a noncomputable c.e. many-one degree containing a D -maximal set D .

If D is of Type 1, then D is simple, so Maslova's theorem applies. If D is of Type 2, the proof of [Theorem 4.1](#) gives a simple set $S \equiv_m D$, and Maslova's theorem applied to S yields a least finite-one degree inside $\mathbf{d}$.

Suppose finally that D is of one of Types 3–10. Let $B \equiv_m D$. Then B is noncomputable, and in particular $B \leq_m D$. By [Theorem 1.2](#),

$$D \leq_1 B,$$

so $D \leq_{fo} B$. Since $B \equiv_m D$ was arbitrary, D itself represents the least finite-one degree inside $\mathbf{d}$. $\square$

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The first version of this paper was developed with extensive use of ChatGPT Pro (OpenAI) and Gemini Deep Think (Google DeepMind) in exploratory analysis and technical verification. In preparing the present substantially revised version, ChatGPT with the GPT-5.6 Sol model (OpenAI) was used primarily for technical checking and manuscript preparation. The author has reworked and verified the arguments presented here and bears sole responsibility for the mathematical content of this version.

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